Math 317 Winter 2014 Homework 3 Solutions

Due Feb. 5 2p

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answers.

Question 1. *Calculate the radius of convergence of the power series*

$$\sum_{n=1}^{\infty} \left(\frac{3}{n} + \frac{6}{n^2}\right) x^n. \tag{1}$$

Solution. We have

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} \left(\frac{3}{n} + \frac{6}{n^2}\right)^{1/n}. \tag{2}$$

Note that

$$\left(\frac{3}{n}\right)^{1/n} < \left(\frac{3}{n} + \frac{6}{n^2}\right)^{1/n} < \left(\frac{9}{n}\right)^{1/n} \tag{3}$$

thus by Squeeze Theorem we conclude

$$\lim_{n\to\infty} \left(\frac{3}{n} + \frac{6}{n^2}\right)^{1/n} = 1 \Longrightarrow \limsup_{n\to\infty} \left(\frac{3}{n} + \frac{6}{n^2}\right)^{1/n} = 1 \tag{4}$$

therefore the radius of convergence is 1.

Question 2. *Find all $x \in \mathbb{R}$ where the series*

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} e^{nx} \tag{5}$$

converges.

Solution. For any $x \in \mathbb{R}$, we notice that,

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} e^{nx} \text{ converges at } x \in \mathbb{R} \Longleftrightarrow \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} y^n \text{ converges at } y = e^x. \tag{6}$$

The radius of convergence of the latter (power) series is 1. Furthermore, at $y = 1$, we have

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} y^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\ln n} \text{ converges;} \tag{7}$$

while at $y = -1$, we have

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} y^n = \sum_{n=2}^{\infty} \frac{1}{\ln n} \text{ diverges.} \tag{8}$$

Thus $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} y^n$ converges if and only if $-1 < y \leqslant 1$. Consequently $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} e^{nx}$ converges if and only if $-1 < e^x \leqslant 1$ which gives $-\infty < x \leqslant 0$.

Question 3. *Consider the infinite series in Question 2.*

a) *Discuss the uniform convergence of the series.*

b) *Is the sum a continuous function (meaning: continuous at every x where it is defined)?*

Solution.

a) We denote

$$f(y) := \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} y^n \tag{9}$$

which is defined for $-1 < y \leqslant 1$. Then from a),

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} e^{nx} = g(x) := f(e^x). \tag{10}$$

By Abel's theorem, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} y^n$ converges uniformly on $[0,1]$. Thus for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\forall y \in [0,1], \quad \forall m > N, \qquad \left| \sum_{n=2}^{m} \frac{(-1)^n}{\ln n} y^n - f(y) \right| < \varepsilon. \tag{11}$$

This gives:

$$\forall x \in (-\infty, 0], \quad \forall m > N, \qquad \left| \sum_{n=2}^{m} \frac{(-1)^n}{\ln n} e^{nx} - g(x) \right| < \varepsilon. \tag{12}$$

Thus $\sum_{n=0}^{\infty} \frac{(-1)^n}{\ln n} e^{nx}$ converges uniformly on $(-\infty, 0]$.

b) As each $\frac{(-1)^n}{\ln n} e^{nx}$ is continuous on $(-\infty, 0]$, the uniform convergence of the series implies that the sum $g(x)$ is continuous on $(-\infty, 0]$. Thus $g(x)$ is a continuous function.

Question 4.

a) *Prove*

$$\forall x \in (-1,1), \qquad \frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}. \tag{13}$$

b) *Then prove*

$$\forall x \in (-1,1), \qquad \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}. \tag{14}$$

Solution.

a) We prove by definition. (Recall that the RHS being the Taylor expansion of the LHS has nothing to do with whether the two sides are equal or not!)

Let $x \in (-1, 1)$ be arbitrary. Let $\varepsilon > 0$ be arbitrary. Take $N > \log_{|x|}\varepsilon$. Then for every $n > N$, we have

$$\left| \frac{1}{1+x^2} - (1 - x^2 + \cdots + (-1)^n x^{2n}) \right| = \left| \frac{x^{2n+2}}{1+x^2} \right| < |x|^N < \varepsilon. \tag{15}$$

Therefore

$$\forall x \in (-1,1), \qquad \frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}. \tag{16}$$

b) Let $x \in (-1,1)$ be arbitrary. By Fundamental Theorem of Calculus Version 1,

$$\arctan x = \int_0^x \frac{1}{1+u^2} \, \mathrm{d}u. \tag{17}$$

On the other hand, as $|x|<1$, the series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ converges uniformly on $[-|x|, |x|]$ and therefore

$$\int_0^x \frac{1}{1+u^2} \mathrm{d}u = \sum_{n=0}^{\infty} \int_0^x (-1)^n u^{2n} \mathrm{d}u = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}. \tag{18}$$

Thus ends the proof.

Question 5. *Without using Abel's theorem, prove directly through the re-summation technique that*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \tag{19}$$

converges uniformly on $[0,1]$*. Then prove*

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots. \tag{20}$$

Solution. We prove that the series is uniformly Cauchy, that is for any $\varepsilon>0$, there is $N \in \mathbb{N}$ such that for every $m>n>N$,

$$\forall x \in [0,1], \qquad \left| \sum_{k=n+1}^{m} \frac{(-1)^k}{2k+1} x^{2k+1} \right| < \varepsilon. \tag{21}$$

Let $\varepsilon>0$ be arbitrary. Since $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ is convergent, there is $N \in \mathbb{N}$ such that for every $m>n>N$,

$$\left| \sum_{k=n+1}^{m} \frac{(-1)^k}{2k+1} \right| < \frac{\varepsilon}{2}. \tag{22}$$

Now for the same n, m we denote

$$S_k := \sum_{l=n+1}^{k} \frac{(-1)^l}{2l+1} \tag{23}$$

which satisfies

$$\forall k \in \{n+1, \ldots, m\}, \qquad \frac{(-1)^k}{2k+1} = S_k - S_{k-1}; \qquad |S_k| < \frac{\varepsilon}{2}. \tag{24}$$

Here we set $S_n := 0$.

Now we calculate

$$\begin{aligned}
\left| \sum_{k=n+1}^{m} \frac{(-1)^k}{2k+1} x^{2k+1} \right| &= \left| \sum_{k=n+1}^{m} (S_k - S_{k-1}) x^{2k+1} \right| \\
&= \left| (S_{n+1} - S_n) x^{2(n+1)+1} + \cdots + (S_m - S_{m-1}) x^{2m+1} \right| \\
&= \left| S_m x^{2m+1} - S_n x^{2(n+1)+1} + \sum_{l=n+1}^{m-1} S_l (x^{2l+1} - x^{2l+3}) \right| \\
&\leqslant |S_m| + \sum_{l=n+1}^{m-1} |S_l| (x^{2l+1} - x^{2l+3}) \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \left[\sum_{l=n+1}^{m-1} (x^{2l+1} - x^{2l+3}) \right] \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} [x^{2n+3} - x^{2m+1}] \\
&\leqslant \varepsilon.
\end{aligned} \tag{25}$$

where we have taken advantage of $S_n = 0$, $x \in [0,1]$, and for such x, $x^{2l+1} - x^{2l+3} \geqslant 0$.

Now let

$$F(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}. \tag{26}$$

From the above (and Question 4) we know that

i. $F(x)$ is continuous on $[0,1]$;

ii. $F(x) = \arctan x$ on $[0,1)$.

Since $\arctan x$ is continuous at $x = 1$, we must have

$$\frac{\pi}{4} = \arctan 1 = \lim_{x \to 1} \arctan x = \lim_{x \to 1} F(x) = F(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots. \tag{27}$$

Question 6. *Let the radii of convergence for* $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x^n$ *be* R_1, R_2 *respectively.*

a) Prove that the radius of convergence R *for the series* $\sum_{n=0}^{\infty} (a_n b_n) x^n$ *satisfies* $R \geqslant R_1 R_2$.

b) Show through an example that strict inequality may hold: $R > R_1 R_2$.

Note: For part a) you shouldn't assume the existence of any of $\lim_{n\to\infty} |a_n|^{1/n}, \lim_{n\to\infty} |b_n|^{1/n}$, *or* $\lim_{n\to\infty} |a_n b_n|^{1/n}$.

Solution.

a) Since

$$R_1^{-1} = \limsup_{n\to\infty} |a_n|^{1/n}; \quad R_2^{-1} = \limsup_{n\to\infty} |b_n|^{1/n}; \qquad R^{-1} = \limsup_{n\to\infty} |a_n b_n|^{1/n} \tag{28}$$

all we need to prove is the following:

Let $\{x_n\}, \{y_n\}$ be non-negative sequences. Then

$$\limsup_{n\to\infty} (x_n y_n) \leqslant \left(\limsup_{n\to\infty} x_n\right)\left(\limsup_{n\to\infty} y_n\right). \tag{29}$$

By definition

$$\begin{aligned} \limsup_{n\to\infty} (x_n y_n) &= \lim_{n\to\infty} [\sup \{x_n y_n, x_{n+1} y_{n+1}, x_{n+2} y_{n+2}, \ldots\}] \\ &\leqslant \lim_{n\to\infty} [\sup \{x_n, x_{n+1}, \cdots\} \cdot \sup \{y_n, y_{n+1}, \ldots\}] \\ &= \left[\lim_{n\to\infty} (\sup \{x_n, x_{n+1}, \ldots\})\right] \cdot \left[\lim_{n\to\infty} (\sup \{y_n, y_{n+1}, \ldots\})\right] \\ &= \left(\limsup_{n\to\infty} x_n\right)\left(\limsup_{n\to\infty} y_n\right). \end{aligned} \tag{30}$$

b) Take $a_n = [1 + (-1)^n]$ and $b_n = [1 + (-1)^{n+1}]$. Then $a_n b_n = 0$ for all n. Therefore $\infty = R > R_1 R_2 = 1$.