Math 317 Winter 2014 Homework 1

Due Wednesday Jan. 15, 2014 2pm

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answers.

Question 1. *Are the following series convergent or divergent? Justify your answers.*

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n!}}, \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}. \tag{1}$$

Question 2. *Let* $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ *be non-negative series with* $a_n > 0, b_n > 0$ *for all* $n \in \mathbb{N}$. *Further assume that* $\forall n \in \mathbb{N}$, $\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$. *Prove that* $\sum_{n=1}^{\infty} b_n$ *converges* $\Longrightarrow \sum_{n=1}^{\infty} a_n$ *converges.*

Question 3. *Prove by definition, without using improper integrals, that* $\sum_{n=1}^{\infty} \frac{1}{n \log_2 (n+1)} = \infty$.

Question 4. *Prove that if* $\sum_{n=1}^{\infty} a_n$ *is a convergent series of positive numbers, then so is* $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$ *(Note that this gives another proof of the fact that there can be no "largest" convergent series) (Hint: [1])*

Question 5. *Let* $a_n > 0$. *Assume that* $\sum_{n=1}^{\infty} a_n$ *diverges. Prove that* $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ *also diverges.*

Question 6. *Assume* $\sum_{n=1}^{\infty} a_n$ *converges. Prove that* $\sum_{n=1}^{\infty} \frac{a_n}{n}$ *must also converge.*

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1. *Apply Young's inequality to obtain* $a_n^{n/(n+1)} \leqslant C_1 a_n + C_2 b_n$.