Math 317 Winter 2014 Final Study Guide

April 9, 2014

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1. Required Sections in Lecture Notes

The following lists minimal requirements for passing the course. To reach better grades you may need to know and understand more. There is no upper bound for how much more.

- Week 1
  - 1.1, 1.2, 1.3; 2.1, 2.2; 3.1; 4.1.
- Week 2
  - 1.1, 1.2; 2.1 (except proof for differentiability), 2.2; 3.1.
- Week 3
  - 2.1, 2.2; 3.1, 3.2.
- Week 4:
  - 2.1, 2.2; 3.1, 3.2 (except proof of the Theorem), 3.3.
- Week 5
  - 1.1, 1.2 (except proof of the S-B Theorem); 2.1, 2,2; 3.1, 3.2, 3.3.
- Week 6
  - 1.1, 1.2, 1.3;
  - Some working knowledge of ordinal numbers, as involved in homeworks and quizzes.
- Week 7
  - 1.1, 1.2.2, 2.2, 3.2;
  - Proof of Green's Theorem for unit square; Apply Green's Theorem to calculation.
- Week 8
  - 1.1, 1.2; 2.1, 2.2; 3.1, 3.2, 3.3.
- Week 9
  - 1.1, 1.2; 2.1, 2.2; 3.1, 3.2.

Plus: All the exercises in Section 5.1 of lecture notes for Weeks 1 – 9.
You should also review all Homework and Quiz problems.

2. Infinite Series

- Infinite series (of numbers).
  - Key concept: Convergence/divergence.
  - To prove convergence/divergence:
    1. Check $\lim_{n\to\infty} a_n \overset{?}{=} 0$;
    2. Try ratio/root tests;
    3. Comparison;
    4. Definition/Cauchy criterion.

    If 1–4 fail, try re-summation.
  - Main examples: Geometric series; Generalized harmonic series (p-series); Alternating series.
- Infinite series of functions
  - Key concepts:
    - Convergence at a particular point $x = c$: $\sum_{n=1}^{\infty} u_n(c)$ is just an infinite series of numbers.
    - Uniform convergence: For $\sum_{n=1}^{\infty} u_n(x)$ be as "good" as each $u_n(x)$.[1]
  - To prove uniform convergence on $[a, b]$:
    1. M-test;
       a. Inspect each $u_n(x)$ and try to find a convergent series of numbers $\sum a_n$ such that $|u_n(x)| \leqslant a_n$ for all $x \in [a, b]$ – note the absolute value!
       b. If that is hard to do, then try to solve $\max_{x\in[a,b]} |u_n(x)|$ and set the maximum as a_n.

       Exercise 1. Let $\sum_{n=1}^{\infty} u_n(x)$ be such that $\sup_{x\in[a,b]} |u_n(x)| = 1$ for all n. Can it be uniformly convergent on $[a, b]$? (Ans:[2])

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1. Uniform convergence of $\sum u_n(x)$ itself is enough for continuity and integrability, but uniform convergence of $\sum u_n'(x)$ is also required to guarantee differentiability of the sum.

2. No. Assume otherwise, that is $\sum u_n(x) \longrightarrow f$ uniformly. Then there is $N \in \mathbb{N}$ such that for all $n > N$,

$$\forall x \in [a, b], \qquad \left| \sum_{k=1}^{n} u_k(x) - f(x) \right| < \frac{1}{3}. \tag{1}$$

This gives

$$\forall x \in [a, b], \qquad |u_{N+2}(x)| = \left| \left(\sum_{k=1}^{N+2} u_k(x) - f(x) \right) - \left(\sum_{k=1}^{N+1} u_k(x) - f(x) \right) \right| < \frac{2}{3} \tag{2}$$

which contradicts $\sup_{x\in[a,b]} |u_{N+2}(x)| = 1$.

Exercise 2. Let $\sum_{n=1}^{\infty} u_n(x)$ be such that $\sup_{x\in[a,b]} |u_n(x)| = 1/n$ for all n. Can it be uniformly convergent on $[a,b]$? (Ans:[3])

2. Definition/Cauchy.

- Special infinite series of functions
  - ○ Power series: $u_n(x) = a_n\,(x - x_0)^n$ for some x_0.
    - – Source: Taylor expansion; Solutions to ODEs.
    - – Key concept: radius of convergence.
      - Inside: uniform convergence;[4]
      - Outside: divergence;
      - On the boundary: different for each power series.
    - – Calculation formula is based on root test.
    - – The first thing you do whenever given a power series should be calculating its radius of convergence, since once this is done you only need to focus on two values of x.
  - ○ Fourier series.
    - – Source: Solutions to PDEs (Separation of variables).
    - – Key concepts:
      - Periodic function and periodic extension.
      - Orthogonality relation $\Longrightarrow$ formulas to calculate the expansion;
      - Convergence of Fourier expansion of piecewise smooth functions.
      - Dirichlet kernels: Semi-explicit formula for the partial sums, makes study of convergence possible.

Example 1. Let $a \in \mathbb{R}$. Study the convergence of

$$1 - \frac{x}{2^a} - \frac{x^2}{3^a} + \frac{x^3}{4^a} + \frac{x^4}{5^a} - \frac{x^5}{6^a} + \frac{x^6}{7^a} - \frac{x^7}{8^a} - \frac{x^8}{9^a} + \frac{x^9}{10^a} + \frac{x^{10}}{11^a} - \frac{x^{11}}{12^a} + \cdots \tag{3}$$

where the pattern of the signs are $+, -, -, +, +, -$ with period 6.

Solution. We observe that this is a power series.

- First calculate radius of convergence – note that the r.o.c. formula has absolute value and therefore disregards the sign pattern which can take some time to condense into one formula.

$$\lim_{n\to\infty} \left|\frac{1}{n^a}\right|^{1/n} = \lim_{n\to\infty} n^{-a/n} = 1 \Longrightarrow R = 1. \tag{4}$$

3. Yes. For example $u_n(x) = \frac{(-1)^n}{n}$.

4. Not only for $\sum u_n(x)$, but also for $\sum u_n'(x)$, $\sum u_n''(x)$, ...

Therefore no matter what a is, the series

- converges on $(-1,1)$, uniformly on $[-a,a]$ for every $a\in(0,1)$;
- diverges on $\{|x|>1\}$.

Thus we see that all we need to seriously study are the two cases $x=\pm 1$.

- Now notice that when $x=\pm 1$, we have

$$\left|\pm\frac{x^{n-1}}{n^a}\right|=\frac{1}{n^a}. \tag{5}$$

Therefore again, without any study of the sign pattern, we conclude: The series

  - converges at both $x=\pm 1$ when $a>1$, since $\sum \frac{1}{n^a}$ converges;
  - diverges at both $x=\pm 1$ when $a\leqslant 0$, since the generic term does not $\rightarrow 0$.

- Finally we study the case $a\in(0,1]$ and $x=\pm 1$. This is when we finally need to understand a bit the sign pattern.

  - $x=1$. In this case we write the generic term of the series as $a_n=s_n\frac{1}{n^a}$ where s_n takes value ± 1. Following the observation

$$\forall K\in\mathbb{N},\qquad \sum_{n=1}^{6K} s_n=0 \tag{6}$$

we easily see that

$$\left|\sum_{n=1}^{N} s_n\right|\leqslant 1 \tag{7}$$

for all $N\in\mathbb{N}$.

Exercise 3. Let $a\in(0,1]$. Prove that the series converges at $x=1$. (Hint:[5])

  - $x=-1$. Since $2|\,6$, the sign pattern of the new series still have 6 as its period, and within one period the pattern is $+,+,-,-,+,+$.

Exercise 4. Does Abel's re-summation still work here? (Hint:[6])

We notice that in each period:

$$\frac{1}{(6k+1)^a}+\frac{1}{(6k+2)^a}-\frac{1}{(6k+3)^a}-\frac{1}{(6k+4)^a}+\frac{1}{(6k+5)^a}+\frac{1}{(6k+6)^a}\geqslant\frac{2}{(6k+6)^a}. \tag{8}$$

Exercise 5. Prove that the series diverges at $x=-1$ when $a\in(0,1]$. (Hint:[7])

Remark 2. If you have some training in probability, try study the convergence/divergence of $\sum_{n=1}^{\infty}\frac{s_n}{n}$ where s_n is a random variable (say takes $1,-1$ with equal probability).

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5. Abel.

6. Each period sum up to 2. Thus $\sum_{n=1}^{N} s_n$ grows like $N/3$ and is thus not bounded. So Abel's Theorem does not apply anymore. However this does not mean we cannot resolve the convergence issue through estimating the re-summed series $\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} s_k\right)\left(\frac{1}{n^a}-\frac{1}{(n+1)^a}\right)$...

7. The partial sum of the first $6K$ terms goes to ∞.

3. Cardinals and Ordinals

- Comparing sizes of sets:
  - $f: A \mapsto B$ one-to-one $\Longrightarrow A \lesssim B$;
  - $g: A \mapsto B$ onto $\Longrightarrow A \gtrsim B$;
  - $h: A \mapsto B$ bijection (one-to-one + onto) $\Longrightarrow A \sim B$.
- Proving $A \sim B$:

  Always try to prove $A \lesssim B$ and $B \lesssim A$ and then apply Schroeder-Bernstein, unless the problem forbids that approach.
- Basic cardinality relations:

$$\mathbb{N} \sim \mathbb{Q} \sim \mathbb{N}^k, \quad k \in \mathbb{N}; \tag{9}$$

$$\mathbb{N}^{\mathbb{N}} \sim 2^{\mathbb{N}} \sim \mathbb{R} \sim \mathbb{R}^k \sim \mathbb{R}^{\mathbb{N}}. \tag{10}$$

- Orders: Partial, total/linear, well. Examples for each of the three.
- Know how to prove/disprove for each type of ordering.
- Re-order sets to have a particular order-type.
- In particular, re-order sets to get a particular ordinal number.

  Note. As ordinal numbers are special well-ordered sets, it does not make sense to talk about the ordinal number for a certain set if it is not well-ordered.

Example 3. Let $A := \left\{ f: [0, 2\,\pi] \mapsto \mathbb{R} |\, f \text{ is integrable; } \exists \text{a trigonometric series such that } \int_0^{2\pi} \left| f - \left\{ \frac{a_0}{2} + \sum_{n=1}^N [a_n \cos n\,x + b_n \sin n\,x] \right\} \right|^2 \longrightarrow 0 \text{ as } N \longrightarrow \infty \right\}$. Find the cardinality of A.

Solution. The idea is to compare A with

$$B := \{\{a_0, a_1, \ldots, a_n, \ldots; b_1, \ldots, b_n, \ldots\} |\, a_n, b_n \in \mathbb{R}\} \sim \mathbb{R}^{\mathbb{N}} \sim \mathbb{R}. \tag{11}$$

Since for now we do not have a good characterization of B[8] we can only expect to show $A \lesssim B$. Thus we need to find a one-to-one mapping $F: A \mapsto B$.

8. Which is in fact $\sum_{n=1}^{\infty} (a_n^2 + b_n^2) < \infty$.

Naturally we define

$$F(f) := \{a_0, a_1, \ldots, a_n, \ldots; b_1, \ldots, b_n, \ldots\} \tag{12}$$

if $\int_0^{2\pi} \left| f - \left\{\frac{a_0}{2} + \sum_{n=1}^N [a_n \cos n\, x + b_n \sin n\, x]\right\}\right|^2 \longrightarrow 0$ as $N \longrightarrow \infty$. To see that it is one-to-one, let f, g be such that $\int_0^{2\pi} |f - g|^2 \,\mathrm{d}x = \varepsilon_0 > 0$.[9] Now we calculate

$$\int_0^{2\pi} \left| f - \left\{\frac{a_0}{2} + \sum_{n=1}^N [a_n \cos n\ x + b_n \sin n\ x]\right\}\right|^2 + \left| g - \left\{\frac{a_0}{2} + \sum_{n=1}^N [a_n \cos n\ x + b_n \sin n\ x]\right\}\right|^2 \geqslant \frac{1}{2}\int_0^{2\pi} |f - g|^2 \mathrm{d}x \tag{13}$$

where we have used the simple fact $a^2 + b^2 \geqslant \frac{1}{2}(a - b)^2$ for any number $a, b \in \mathbb{R}$. Therefore $F(f) \neq F(g)$ and the mapping is one-to-one.

Summarizing, we have proved

$$A \lesssim B \sim \mathbb{R}. \tag{14}$$

On the other hand, consider $G: \mathbb{R} \mapsto A$ defined as

$$G(a_0) = \frac{a_0}{2}. \tag{15}$$

Then clearly G is one-to-one. Consequently $\mathbb{R} \lesssim A$. The conclusion is thus a consequence of Schröder-Bernstein: $A \sim \mathbb{R}$.

Example 4. Re-order $\mathbb{Z}$ to be similar to $\mathbb{Q}$.

Solution. Note that this can be done as soon as we can re-order $\mathbb{N}$ to be similar to $\mathbb{Q}^+ := \mathbb{Q} \cap \{x > 0\}$. To do this, let $f: \mathbb{N} \mapsto \mathbb{Q}^+$ be a bijection (such as the one from listing $\mathbb{Q}^+$ as a $\infty \times \infty$ matrix. Then we define the new order on $\mathbb{N}$ as

$$x < y \Longleftrightarrow f(x) < f(y). \tag{16}$$

Remark 5. We see that "re-ordering" A to be similar to B is in fact the construction of a bijection $F: A \mapsto B$ and then define the new ordering for A as

$$x <_{A,\text{new}} y \Longleftrightarrow F(x) <_B F(y). \tag{17}$$

Exercise 6. Re-order $\mathbb{Q}$ to be similar to $[0, 1) \cap \mathbb{Q}$.

Example 6. Re-order $\mathbb{Z}$ to get $\omega \cdot 2 + 3$.

Solution. $2 < 3 < \cdots < -2 < -3 < \cdots < 0 < 1 < -1$.

9. So we are being a bit sloppy here. At the end we are in fact calculating the cardinality of the quotient set $A/\approx$ where $f \approx g \Longleftrightarrow \int\ |f - g|^2 \,\mathrm{d}x = 0$. Sorry that I didn't realize this in today's lecture.

4. Vector Calculus

- Concepts: Line integrals of 1st and 2nd kind; Surface integrals of 1st and 2nd kind.
  - Physical interpretations:
    - Integrals of the 1st kind: Mass of the curve/surface;
    - Integrals of the 2nd kind: Flow past the curve/surface.
- Calculation:
  - Line integral:

$$\text{Line integral} \xrightarrow{\text{''pull-back'' via parametrization}} \text{single variable integral;} \tag{18}$$

  - Surface integral:

$$\text{Surface integral} \xrightarrow{\text{''pull-back'' via parametrization}} \text{double integral.} \tag{19}$$

Remark 7. (Some intuition of the calculation formula)
Imagine we try to estimate how much of a certain substance is being carried by the wind (or river flow) past a certain place. What we could do is to hang a net and say at each node put a sensor. The net would be deformed by the hanging points and also by the wind, thus becoming a curved surface.

After a while we take the net down and spread it on the flat ground ("pull-back" from the (x, y, z) space via parametrization to the parameter (u, v) plane). Now we need to multiply the reading of each sensor by the ratio between the area it represents and the actual area of the sensor (integrate with respect to (u, v)). But the "area it represents" should not be the area when the net is on the ground, but should be the area when it is hung in the wind. Thus we need to further multiply by a "deformation factor": $\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|$.

- Theorems: Green; Stokes; Gauss.

$$\text{Green: Line integral} \longleftrightarrow \text{Area integral;} \tag{20}$$

$$\text{Stokes: Line integral} \longleftrightarrow \text{Surface integral;} \tag{21}$$

$$\text{Gauss: Surface integral} \longleftrightarrow \text{Volume integral} \tag{22}$$

Note. The "line integral" and "surface integral" above are integrals of the 2nd kind so need to pay attention to their orientations.

  - Know how to prove in simplest situations (see lecture notes);
  - Know the relations between the theorems;

Example 8. (Toricelli's trumpet/Gabriel's horn) Consider the part of the curve $y = 1/x$ when $x \geqslant 1$. Let S be the surface obtained from rotating this curve around the x-axis (thus it has the shape of a trumpet). Let V be the volume "inside" this trumpet. Then the area of S is infinite but the volume of V is finite.[10]

Solution. First notice that every point on the surface S is obtained by rotating a point on the curve $y=1/x$. Thus we can parametrize S by: $\left(u, \frac{1}{u}\cos v, \frac{1}{u}\sin v\right)$ with $D=\{(u,v)|\, u\geqslant 1, 0\leqslant v<2\,\pi\}$.

Now calculate

$$\boldsymbol{r}_u=\begin{pmatrix} 1 \\ -\frac{\cos v}{u^2} \\ -\frac{\sin v}{u^2} \end{pmatrix}, \qquad \boldsymbol{r}_v=\begin{pmatrix} 0 \\ -\frac{\sin v}{u} \\ \frac{\cos v}{u} \end{pmatrix}. \tag{23}$$

We have

$$E=1+u^{-4}, \qquad F=0, \qquad G=u^{-2}\Longrightarrow \|\boldsymbol{r}_u\times\boldsymbol{r}_v\|=\sqrt{E\,G-F^2}=u^{-1}\sqrt{1+u^{-4}} \tag{24}$$

where an absolute value sign has been drop because $u\geqslant 1\Longrightarrow u^{-1}\geqslant 0\Longrightarrow \sqrt{u^{-2}}=u^{-1}$.

Now calculate

$$\int_D \|\boldsymbol{r}_u\times\boldsymbol{r}_v\|\,\mathrm{d}(u,v)=\int_1^\infty\left[\int_0^{2\pi} u^{-1}\sqrt{1+u^{-4}}\,\mathrm{d}v\right]\mathrm{d}u\geqslant 2\,\pi\int_1^\infty u^{-1}\,\mathrm{d}u=\infty. \tag{25}$$

Thus the area is infinite.

On the other hand, application of Fubini leads to

$$\begin{aligned}\text{Volume}(V) &= \int_1^\infty\left[\int_{y^2+z^2\leqslant 1/x^2}\mathrm{d}(y,z)\right]\mathrm{d}x\\ &= \int_1^\infty \frac{\pi}{x^2}\,\mathrm{d}x=\pi.\end{aligned} \tag{26}$$

It is finite.

Exercise 7. Solve the problem using a different parametrization. (Hint:[11])

Problem 1. Let $y=f(x)\geqslant 0$ be bounded on $[1,\infty)$. Let S be the surface obtained from rotating the graph around the x-axis. Let V be the volume "inside" S. Prove:

$$\text{Area}(S) \text{ is finite}\Longrightarrow \text{Vol}(V) \text{ is finite.} \tag{27}$$

(Solution:[12])

10. To fully appreciate the significance of this example, one should know that this was proposed way before Newton and Leibniz developed the theory of Calculus (Torricelli died when Newton was 5). The example led to fierce debates between important figures such as Torricelli, Hobbes, and Galileo.

11. For example consider $\left(\frac{1}{\sqrt{u^2+v^2}}, u, v\right)$ with $D:=\{(u,v)|\, 0<u^2+v^2\leqslant 1\}$.

12. See http://en.wikipedia.org/wiki/Gabriel's_Horn.