Comments on Homework 4

February 27, 2014

1. Mistakes.

The following are popular mistakes. Try to fully understand why they are wrong.

i. Take P_k such that $x_k \notin P_k$ ... By construction $\forall k, x_k \notin \cap_1^\infty \overline{P_k}$.

ii. Since $\lim_{n\to\infty} [\lim_{t\to 0} D_n(t)] = +\infty$,

$$\lim_{n\to\infty} \int_{-\delta}^{\delta} D_n(t)\,\mathrm{d}t = \infty. \tag{1}$$

iii. Since $p(0)^k \longrightarrow \infty$, $\int\ p(x)^k\,\mathrm{d}x \longrightarrow \infty$.

iv. There is $\delta > 0$ such that for every $x \in (-\delta, \delta)$, $\cos(m\,x) > 0$ for all $m \in \mathbb{N} \cup \{0\}$.

v. Forgot to discuss $n = 2$ separately in Question 3.

vi. Forgot to justify $f(x) =$ its Fourier expansion at $x = \frac{\pi}{2}$ in Question 1.

vii. Since f is continuous at x_0, there is $\delta > 0$ such that f is continuous on $(x_0 - \delta, x_0 + \delta)$.

viii. Let $f(x)$ have Taylor expansion $\sum_{n=0}^\infty a_n (x - x_0)^n$ at $x_0 \in \mathbb{R}$. Assume that the power series $\sum_{n=0}^\infty a_n (x - x_0)^n$ has radius of convergence $R > 0$, then

$$f(x) = \sum_{n=0}^\infty a_n (x - x_0)^n \qquad \text{on } (x_0 - R, x_0 + R). \tag{2}$$

ix. P is countable and therefore can be listed as $\{x_1, x_2, \ldots., x_n\}$.

x. Take $I_1, I_2, \ldots.$ open intervals... S is compact, then $S \cap I_k$ is compact ...

xi. Question 5: It suffices to prove convergence at $x = 0$. [1]

2. Exercises.

Some related exercises.

Exercise 1. Find a function $f(x) > 0$ such that $\lim_{x\to 0} f(x) = +\infty$ but $\int_{-1}^{1} f(x)\,\mathrm{d}x < \infty$.

Exercise 2. Let $p(x)$ be continuous and non-negative on $[-1, 1]$. Prove

$$p(0)^k \longrightarrow \infty \Longrightarrow \int_{-1}^{1} p(x)^k\,\mathrm{d}x \longrightarrow \infty. \tag{3}$$

Show that neither hypothesis (continuity, non-negativity) can be dropped. (Hint:[2])

Exercise 3.

a) Find a nested sequence of open intervals $I_1 \supset I_2 \supset I_3 \supset \ldots.$ such that $\cap_{n=1}^\infty I_n = \varnothing$. (Hint:[3])

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1. This is not sufficient for proving uniform convergence.
2. $p(0) = 2, p(x) = 0$.
3. $(0, 1/n)$;

b) Find a nested sequence of compact intervals $I_1 \supset I_2 \supset I_3 \supset \ldots.$ such that $\cap_{n=1}^{\infty} I_n$ is not a single point. (Hint:[4])

Exercise 4. Find a compact set S and an open interval I such that $S \cap I$ is not compact. (Hint:[5])

Exercise 5. Find a function $f(x)$ such that

i. it is continuous at $x = 0$;

ii. For any $\delta > 0$, it is not continuous on $(-\delta, \delta)$.

(Hint:[6])

Exercise 6. Let

$$f(x) = \begin{cases} \exp\left[-\frac{1}{x^2}\right] & x \neq 0 \\ 0 & x = 0 \end{cases}. \tag{4}$$

Prove that $f(x)$ has Taylor expansion at $x_0 = 0$ and furthermore this power series has radius of convergence ∞. Nevertheless $f(x)$ equals this power series only at $x = 0$. (Hint:[7])

3. Other comments.

- $f(x) > 1 \Longrightarrow \int f(x)^k \,\mathrm{d}x \longrightarrow \infty$.
  This is not incorrect, but looks like hard to prove without using Lebesgue integration theory (unless assume f is continuous somewhere, of course).

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4. $[1 - 1/n, 2 + 1/n]$.
5. $S = \{0\} \cup \{1/n\}$.
6. $f(x) = x^2$ when x is rational and 0 otherwise.
7. $f^{(n)}(0) = 0$ for all n.