Math 314 Fall 2013 Homework 3 Solutions

DUE WEDNESDAY OCT. 2 5PM IN ASSIGNMENT BOX (CAB 3RD FLOOR)

- There are 6 problems, each 5 points. Total 30 points.
- Please justify all your answers through proof or counterexample.

Question 1. *Let* $x_n = (-1)^n - e^{-n}$ *and* $E = \{x_n : n \in \mathbb{N}\}$. *(*$\mathbb{N} = \{1, 2, 3, \ldots\}$*). Find* $\max E, \sup E, \min E, \inf E$. *Justify your answers.*

Solution.

- $\max E$ does not exist. Assume the contrary, that is $x_{n_0} = \max E$. Then we have $x_{n_0+2} > x_{n_0}$. Contradiction.
- $\sup E = 1$. Since $1 \geqslant (-1)^n \geqslant (-1)^n - e^{-n}$ for all $n \in \mathbb{N}$, 1 is a upper bound. Now for any upper bound b, we have

$$b \geqslant (-1)^{2k} - e^{-2k} = 1 - e^{-2k}. \tag{1}$$

  for all $k \in \mathbb{N}$. Taking limit $k \longrightarrow \infty$, by comparison theorem we have $b \geqslant 1$.

- $\min E = x_1 = -1 - e^{-1}$. We have

$$x_1 = -1 - e^{-1} \leqslant (-1)^n - e^{-n} \tag{2}$$

  for all $n \in \mathbb{N}$ since $-1 \leqslant (-1)^n, e^{-1} \geqslant e^{-n} \Longrightarrow -e^{-1} \leqslant -e^{-n}$.

- Since $\min E$ exists, $\inf E = \min E = -1 - e^{-1}$.

Question 2. *Let* $f : X \mapsto Y$ *be a function. Let* $A, B \subseteq X$ *and* $S, T \subseteq Y$.

a) *Prove: If* $A \subseteq B$ *then* $f(A) \subseteq f(B)$.

b) *Prove: If* $S \subseteq T$ *then* $f^{-1}(S) \subseteq f^{-1}(T)$.

c) *Is it true that* $A \subset B$ *implies* $f(A) \subset f(B)$*? Justify your answer.*

d) *Is it true that* $S \subset T$ *implies* $f^{-1}(S) \subset f^{-1}(T)$*? Justify your answer.*

Proof.

a) To show $f(A) \subseteq f(B)$ all we need to do is to show that every $y \in f(A)$ also belongs to $f(B)$. Take an arbitrary $y \in f(A)$. By definition of $f(A)$ there is $x \in A$ such that $y = f(x)$. Since $A \subseteq B$ we have $x \in B$. Therefore $y \in f(B)$, by definition of the image $f(B)$.

b) Take an arbitrary $x \in f^{-1}(S)$. Then by definition $f(x) \in S \subseteq T$. So $f(x) \in T \Longrightarrow x \in f^{-1}(T)$, by definition of the pre-image f^{-1}.

c) No. Because f may not be one-to-one. Let $f : \mathbb{N} \mapsto \mathbb{N}$ be defined as $f(n) = 1$ for all $n \in \mathbb{N}$. Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$. Then $A \subset B$ but $f(A) = \{1\} = f(B)$.

d) No. Because f may not be onto. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be $f(x) = \sin x$. Take $S = \{y \in \mathbb{R} : -1 \leqslant y \leqslant 1\}$ and $T = \{y \in \mathbb{R} : -2 \leqslant y \leqslant 2\}$. Then $S \subset T$ but $f^{-1}(S) = \mathbb{R} = f^{-1}(T)$. □

Question 3. *Let* $A \subseteq X, B \subseteq Y$ *and* $f : X \mapsto Y$. *Prove that*

a) $f(f^{-1}(B)) \subseteq B$.

b) $f^{-1}(f(A)) \supseteq A$.

c) *If* $B \subseteq f(X)$, *then* $f(f^{-1}(B)) = B$.

Proof.

a) Take an arbitrary $y \in f(f^{-1}(B))$. By definition of the image $f(\cdot)$ there is $x \in f^{-1}(B)$ such that $y = f(x)$. By definition we have $x \in f^{-1}(B)$ means $y = f(x) \in B$. Therefore $y \in f(f^{-1}(B)) \Longrightarrow y \in B$ so a) is proved.

b) Take an arbitrary $x \in A$. Then $f(x) \in f(A)$. By definition of the pre-image f^{-1} we have $x \in f^{-1}(f(A))$. So $A \subseteq f^{-1}(f(A))$.

c) In a) we already proved $f(f^{-1}(B)) \subseteq B$, so to show $f(f^{-1}(B)) = B$ we need to prove further $B \subseteq f(f^{-1}(B))$. Take any $y \in B$. Since $B \subseteq f(X)$ we have $y \in f(X)$. Then there is $x \in X$ such that $f(x) = y$. By definition of f^{-1}, $f(x) = y \in B \Longrightarrow x \in f^{-1}(B)$. This means $y = f(x) \in f(f^{-1}(B))$. So $y \in B \Longrightarrow y \in f(f^{-1}(B))$ which means $B \subseteq f(f^{-1}(B))$. □

Question 4. *Let $x_n = n^a$ for $a \in \mathbb{R}$. Discuss whether $\lim_{n \longrightarrow \infty} n^a$ exists or not. If it exists find and prove the limit. If it does not prove that the limit does not exist.*

Solution. $\lim_{n \longrightarrow \infty} x_n = 0$ when $a < 0$, $= 1$ when $a = 0$, $= +\infty$ when $a > 0$.

- $a < 0$. For any $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $N > \varepsilon^{1/a}$. Then we have for all $n > N$
$$|x_n - 0| = n^a < N^a < \varepsilon. \tag{3}$$
- $a = 0$. In this case $x_n = 1$ for all n. For any $\varepsilon > 0$, take $N = 1$. Then for all $n > N$,
$$|x_n - 1| = |1 - 1| = 0 < \varepsilon. \tag{4}$$
- $a > 0$. For any $M \in \mathbb{R}$, take $N > |M|^{1/a}$. Then for all $n > N$,
$$x_n = n^a > N^a = |M| \geqslant M. \tag{5}$$

Question 5. *Calculate the limits of the following sequences:*
$$x_n = \frac{100\,n^2 - 2\,n^4}{n^4 + 3\,n}, \quad y_n = \sqrt{n+1} - \sqrt{n-1}, \quad z_n = \frac{\sin n^3}{n}. \tag{6}$$
You can use the results from the previous problem.

Solution.

- x_n. We have
$$x_n = \frac{100\,n^{-2} - 2}{1 + 3\,n^{-3}}. \tag{7}$$
We have $100\ n^{-2} \longrightarrow 0$, $-2 \longrightarrow -2$, $1 \longrightarrow 1$, $3\ n^{-3} \longrightarrow 0$ as $n \longrightarrow \infty$. Furthermore we have $1 + 3\,n^{-3} \longrightarrow 1 \neq 0$. Therefore the limit of the ratio exists and is
$$\lim_{n \longrightarrow \infty} x_n = \frac{\lim_{n \longrightarrow \infty} (100\,n^{-2} - 2)}{\lim_{n \longrightarrow \infty} (1 + 3\,n^{-3})} = -2. \tag{8}$$
- y_n. Write
$$y_n = \frac{(\sqrt{n+1} - \sqrt{n-1})\,(\sqrt{n+1} + \sqrt{n-1})}{(\sqrt{n+1} + \sqrt{n-1})} = n^{-1/2}\,\frac{2}{\sqrt{1+n^{-1}} + \sqrt{1-n^{-1}}}. \tag{9}$$
As $\sqrt{1+n^{-1}} + \sqrt{1-n^{-1}} \longrightarrow 2 \neq 0$, the limit of the ratio exists. Therefore we have
$$\lim_{n \longrightarrow \infty} y_n = \left(\lim_{n \longrightarrow \infty} n^{-1/2}\right)\left(\lim_{n \longrightarrow \infty} \frac{2}{\sqrt{1+n^{-1}} + \sqrt{1-n^{-1}}}\right) = 0 \cdot 2 = 0. \tag{10}$$

Remark. For y_n there are many different ways to prove. For example, one can write
$$|y_n| < \frac{1}{\sqrt{n-1}}. \tag{11}$$

Now $\forall \varepsilon > 0$, take $N \geqslant \frac{1}{\varepsilon^2} + 1$, then for all $n > N$, we have $|y_n| < \frac{1}{\sqrt{n-1}} < \frac{1}{\sqrt{N-1}} \leqslant \varepsilon$. Note that we cannot write "take $N = \frac{1}{\varepsilon^2} + 1$" since $1/\varepsilon^2$ may not be a natural number.

- z_n. We have for all $n \geqslant 1$,
$$-\frac{1}{n} \leqslant \frac{\sin n^3}{n} \leqslant \frac{1}{n} \tag{12}$$
By Squeeze Theorem
$$\lim_{n \longrightarrow \infty} \frac{\sin n^3}{n} = 0. \tag{13}$$

Question 6. *Let $0 < y_1 < x_1$ and set*
$$x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \sqrt{x_n y_n}, \qquad n \in \mathbb{N}. \tag{14}$$

a) *Prove that $0 < y_n < x_n$ for all $n \in \mathbb{N}$;*

b) *Prove that y_n is increasing and bounded above, and x_n is decreasing and bounded below;*

c) *Prove that $0 < x_{n+1} - y_{n+1} < (x_1 - y_1)/(2^n)$ for all $n \in \mathbb{N}$;*

d) *Prove that $\lim_{n \longrightarrow \infty} x_n$, $\lim_{n \longrightarrow \infty} y_n$ both exist are are equal.*

Proof.

a) We prove by mathematical induction. Let $P(n)$ be the statement $0 < y_n < x_n$.
- $P(1)$ holds by assumption.
- Suppose $P(n)$ holds, we try to show $P(n+1)$ also holds.
  As $x_n, y_n > 0$, we have $x_{n+1} = \frac{x_n + y_n}{2}$, $y_{n+1} = \sqrt{x_n y_n} > 0$; Next compute
$$x_{n+1} - y_{n+1} = \frac{x_n + y_n - 2\sqrt{x_n y_n}}{2} = \frac{(\sqrt{x_n} - \sqrt{y_n})^2}{2} \geqslant 0. \tag{15}$$
  Since $x_n > y_n$, $\sqrt{x_n} > \sqrt{y_n}$ therefore $x_{n+1} - y_{n+1} > 0$.

b) As $x_n > y_n$, clearly $y_{n+1} = \sqrt{x_n y_n} > \sqrt{y_n^2} = y_n$ and $x_{n+1} = \frac{x_n + y_n}{2} < \frac{x_n + x_n}{2} = x_n$. For the bounds, combining the facts: y_n increasing, $0 < y_n < x_n$, x_n decreasing, we reach
$$y_1 \leqslant y_n < x_n \leqslant x_1 \tag{16}$$
so y_1 and x_1 are the lower/upper bounds respectively.

c) $0 < x_{n+1} - y_{n+1}$ has already been shown in a). For the other inequality, following the argument in a) we have
$$x_{n+1} - y_{n+1} = \frac{(\sqrt{x_n} - \sqrt{y_n})^2}{2} < \frac{(\sqrt{x_n} - \sqrt{y_n})(\sqrt{x_n} + \sqrt{y_n})}{2} = \frac{x_n - y_n}{2}. \tag{17}$$
This gives
$$x_{n+1} - y_{n+1} < \frac{x_n - y_n}{2} < \frac{x_{n-1} - y_{n-1}}{4} < \cdots < \frac{x_1 - y_1}{2^n}. \tag{18}$$

Remark. One student in Fall 2012 Math 314 discovered a much more clever argument. Since y_n is increasing, we have
$$x_{n+1} - y_{n+1} < x_{n+1} - y_n = \frac{x_n + y_n}{2} - y_n = \frac{x_n - y_n}{2}. \tag{19}$$

d) As x_n is decreasing and bounded below by 0, there is $a \in \mathbb{R}$ such that $x_n \longrightarrow a$. On the other hand, we have
$$y_n < x_n < x_1 \tag{20}$$
so $\{y_n\}$ is increasing and bounded above. Thus there is $b \in \mathbb{R}$ such that $y_n \longrightarrow b$. By the monotonicity of x_n and y_n and Comparison theorem, we have
$$y_n \leqslant b \leqslant a \leqslant x_n. \tag{21}$$
This gives
$$a - b \leqslant x_n - y_n < 2^{-(n-1)} |x_1 - y_1|. \tag{22}$$
As this holds for all $n \in \mathbb{N}$, $a = b$. □