Math 314 Fall 2013 Homework 1 Solutions

- There are 6 problems, each 5 points. Total 30 points.
- Please justify all your answers through proof or counterexample.

Question 1. *The Fibonacci numbers are defined through*

$$f_1 = 1,\ f_2 = 1,\ f_3 = 2, \ldots \tag{1}$$

and then through the general formula

$$f_n = f_{n-1} + f_{n-2} \tag{2}$$

for all $n > 2$. Prove using mathematical induction that for all $n > 1$,

$$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}. \tag{3}$$

Proof. Let the statements $P(n)$ be defined as

$$P(n) = \{f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}\} \tag{4}$$

- Prove $P(2)$: When $n = 2$ we have
$$f_1 + f_3 = 3 = f_4. \tag{5}$$
- Prove $P(n) \Longrightarrow P(n+1)$. Assume
$$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}. \tag{6}$$
Adding f_{2n+1} to both sides we have
$$f_1 + f_3 + \cdots + f_{2n-1} + f_{2n+1} = f_{2n} + f_{2n+1} = f_{2n+2} = f_{2(n+1)} \tag{7}$$
which is exactly $P(n+1)$. □

Remark. Note that the claim

$$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}. \tag{8}$$

in fact holds for all $n \geqslant 1$ since $f_1 = f_2$.

Question 2. *Let A, B be mathematical statements. Prove the following*

a) $A \Longrightarrow B$ and $B \Longrightarrow A$ are not equivalent;

b) $A \Longrightarrow B$ and $\neg B \Longrightarrow \neg A$ are equivalent.

Proof. We prove through truth table.

A	B	$A \Longrightarrow B$	$B \Longrightarrow A$	$\neg B \Longrightarrow \neg A$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

(9)

We see that $A \Longrightarrow B$ and $\neg B \Longrightarrow \neg A$ take the same value in all cases while $B \Longrightarrow A$ is different. Therefore $A \Longrightarrow B$ and $\neg B \Longrightarrow \neg A$ are equivalent and $A \Longrightarrow B$ and $B \Longrightarrow A$ are not equivalent. □

Question 3. *Let P be a mathematical statement. If we know that $(\neg P) \Longrightarrow P$ is true, what can we say about P itself?*

Solution. Since there are only two possibilities: P true or P false, we list both:

- P false. Then $\neg P$ is true and $(\neg P) \Longrightarrow P$ is false.

- P true. Then $\neg P$ is false and $(\neg P) \Longrightarrow P$ is true.

Since we know $(\neg P) \Longrightarrow P$ is true, P has to be true.

Question 4. *Let $P(x), Q(x)$ be statements involving a variable x. Critique the following statement:*

$$(\exists x\ P(x)) \wedge (\exists x\ Q(x)) \Longrightarrow [\exists x\ (P(x) \wedge Q(x))]. \tag{10}$$

If it is true, prove it; If it is false, give a counterexample.

Solution. The statement means:

> If there is x such that the statement $P(x)$ holds, and if there is x such that the statement $Q(x)$ holds, then there is x such that both statements hold simultaneously.

This is false. For example take $P(x)$ to be "$x > 1$" and $Q(x)$ to be "$x < 1$". Then both

$$\exists x \quad x > 1 \qquad \text{and} \qquad \exists x \quad x < 1 \tag{11}$$

are true. But the statement

$$\exists x \quad (x > 1) \wedge (x < 1) \tag{12}$$

is clearly false.

Question 5. *Uniform continuity is defined as follows.*

> *A real function $f(x)$ is said to be uniformly continuous if*
>
> $$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \text{ satisfying } |x - y| < \delta, \qquad |f(x) - f(y)| < \varepsilon. \tag{13}$$

Obtain its working negation "f is not uniformly continuous".

Solution. It is

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x, y \text{ satisfying } |x - y| < \delta, \qquad |f(x) - f(y)| \geqslant \varepsilon. \tag{14}$$

Question 6. *The following are facts:*

> *A rainy Tuesday is necessary for a rainy Sunday; If Tuesday rains then Wednesday rains. Wednesday rains only if Friday rains. If Monday is sunny then Friday is sunny; A rainy Monday is sufficient for a rainy Saturday.*

a) *Write the above facts using formal logic statements (Use A — G to denote the statements "Monday rains",...,"Sunday rains".)*

b) *If we know furthermore that it rains on Sunday. Can we say anything about Saturday? Explain.*

Solution.

a) Let the statements A, B, ..., G be "Monday rains", "Tuesday rains", ... "Sunday rains". Then $\neg A$, $\neg B, \ldots, \neg G$ stand for "Monday sunny", "Tuesday sunny", ... "Sunday sunny".
   - A rainy Tuesday is necessary for a rainy Sunday: $G \Longrightarrow B$;
   - If Tuesday rains then Wednesday rains. $B \Longrightarrow C$;
   - Wednesday rains only if Friday rains. $C \Longrightarrow E$;
   - If Monday is sunny then Friday is sunny; $\neg A \Longrightarrow \neg E$. Equivalent to $E \Longrightarrow A$;
   - A rainy Monday is sufficient for a rainy Saturday. $A \Longrightarrow F$.

b) Now we know G. We have

$$[G \wedge (G \Longrightarrow B)] \Longrightarrow B; \tag{15}$$

$$[B \wedge (B \Longrightarrow C)] \Longrightarrow C; \tag{16}$$

$$[C \wedge (C \Longrightarrow E)] \Longrightarrow E; \tag{17}$$

$$[E \wedge (E \Longrightarrow A)] \Longrightarrow A; \tag{18}$$

$$[A \wedge (A \Longrightarrow F)] \Longrightarrow F. \tag{19}$$

Therefore Saturday rains.

Remark. Note that "If" and "Only if" are opposite; "sufficient" and "necessary" are opposite. Therefore, since "B if A", "A is sufficient for B" means $A \Longrightarrow B$, "B only if A", "A is necessary for B" means $B \Longrightarrow A$.