Limit and Continuity

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1. LIMIT OF SEQUENCES

1.1. Definitions and properties.

1.1.1. Definitions.

Definition 1. (Limit $\in\mathbb{R}$) *A sequence of real numbers $\{x_n\}$ is said to* converge *to a real number $a\in\mathbb{R}$ if and only if*

$$\forall\varepsilon>0\ \exists N\in\mathbb{N} \text{ such that } \forall n>N, \qquad |x_n-a|<\varepsilon. \tag{1}$$

We denote this convergence by $x_n\longrightarrow a$, or $\lim_{n\longrightarrow\infty}x_n=a$.

Exercise 1. Suppose we have proved

$$\forall\varepsilon>0\ \exists N\in\mathbb{N} \text{ such that } \forall n>5\,N, \qquad |x_n-a|<20\,\varepsilon, \tag{2}$$

can we conclude $x_n\longrightarrow a$? What if we proved

$$\forall m\in\mathbb{N}\ \exists N\in\mathbb{N} \text{ such that } \forall n>N^2, \qquad |x_n-a|<\frac{1}{\sqrt{m}}, \tag{3}$$

can we conclude $x_n\longrightarrow a$?

Definition 2. (Limit $=\pm\infty$) *A sequence of real numbers $\{x_n\}$ is said to converge to $+\infty$ (also called "diverge to $+\infty$") if and only if*

$$\forall M>0\ \exists N\in\mathbb{N} \text{ such that } \forall n>N, \qquad x_n>M. \tag{4}$$

Denoted $\lim_{n\longrightarrow\infty}x_n=+\infty$ or $x_n\longrightarrow+\infty$ as $n\longrightarrow\infty$.

The definition for $\lim_{n\longrightarrow\infty}x_n=-\infty$ is similar and left as exercise.

Remark 3. Often the + in $+\infty$ is omitted.

Exercise 2. Give precise definition of $\lim_{n\longrightarrow\infty}x_n=-\infty$, and find the working negation of it.

Exercise 3. Suppose we have proved

$$\forall\varepsilon>0\ \exists N\in\mathbb{N} \text{ such that } \forall n>5\,N, \qquad x_n>20\,\varepsilon^{-1}, \tag{5}$$

can we conclude $x_n\longrightarrow+\infty$? What if we proved

$$\forall m\in\mathbb{N}\ \exists N\in\mathbb{N} \text{ such that } \forall n>N^2, \qquad x_n>m^2, \tag{6}$$

What if we proved

$$\forall M\in\mathbb{R}\ \exists N\in\mathbb{N} \text{ such that } \forall n>N, \qquad x_n>|M|? \tag{7}$$

1.1.2. Proving existence and nonexistence of limits.

- Existence. First we make it precise the meaning of "$\{x_n\}$ has a limit in $\mathbb{R}$".

  A sequence of real numbers $\{x_n\}$ has a limit in $\mathbb{R}$ if and only if there is $a\in\mathbb{R}$ such that

  $$\forall\varepsilon>0\ \exists N\in\mathbb{N} \text{ such that } \forall n>N, \qquad |x_n-a|<\varepsilon. \tag{8}$$

  Exercise 4. Obtain the working negation of "$\{x_n\}$ has a limit in $\mathbb{R}$".

  Therefore, to show existence of limit for $\{x_n\}$, we have to do two things:[1]

  - Guess the limit a.

[1]. Later we will see that the theory of Cauchy sequences simplifies the procedure.

- ○ Check that the conditions in the definitions hold.

Example 4. Find the limit of $\left\{\frac{(-1)^n}{n}\right\}$ and justify.
Solution. First we guess that the limit is 0. Now for any $\varepsilon > 0$, let $N > \varepsilon^{-1}$. Then for all $n > N$,

$$\left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon. \tag{9}$$

Exercise 5. What is wrong with the following proof:
First we guess that the limit is 0. Now for any $\varepsilon > 0$, let $N = \varepsilon^{-1}$. Then for all $n > N$,

$$\left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} = \varepsilon. \tag{10}$$

Example 5. Find the limit of $\{\log(n)\}$ and justify.
Solution. First we guess that the limit is $+\infty$. Now for any $M > 0$, take $N > e^M$. Than for all $n > N$,

$$\log(n) > \log(N) > \log(e^M) = M. \tag{11}$$

- Non-existence. To show non-existence, we need to show that
  1. $x_n \not\longrightarrow a$ for any $a \in \mathbb{R}$;
  2. $x_n \not\longrightarrow +\infty$
  3. $x_n \not\longrightarrow -\infty$.

Example 6. (Nonexistence of limit) Show that $\{(-1)^n\}$ has no limit.

Proof. First we show that $x_n \not\longrightarrow a$ for any $a \in \mathbb{R}$. This is usually done through proof of contradiction. Assume the contrary, that is $(-1)^n \longrightarrow a$ for some a. Then for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$, $|(-1)^n - a| < \varepsilon$.

We take $\varepsilon = 1$. There is $N \in \mathbb{N}$ such that for all $n > N$, $|(-1)^n - a| < 1$. Since this holds for all $n > N$, in particular it holds for all odd $n > N$, thus $|-1 - a| < 1$. On the other hand it also holds for all even $n > N$, which leads to $|1 - a| < 1$. Combining the two together we get

$$|-1 - a| + |1 - a| < 2. \tag{12}$$

But on the other hand we have

$$|-1 - a| + |1 - a| = |1 + a| + |1 - a| \geqslant |1 + a + 1 - a| = |2| = 2. \tag{13}$$

Therefore $2 < 2$. Contradiction!

Next we show $x_n \not\longrightarrow +\infty$. We need to show the working negation of the definition:

$$\exists M > 0 \ \forall N \in \mathbb{N} \ \exists n > N \ x_n \leqslant M. \tag{14}$$

For $x_n = (-1)^n$, we simply take $M = 1$. Now for any $N \in \mathbb{N}$, we take $n = N + 1$. Then $x_n = (-1)^{N+1} \leqslant 1$.

The proof for $x_n \not\longrightarrow -\infty$ is similar and left as exercise. □

Exercise 6. Let $\{x_n\}$ be as in the above example. Prove that $x_n \not\longrightarrow -\infty$.

1.2. Properties.

Proposition 7. (Arithmetics) *Let* $a, b \in \mathbb{R}$ $x_n \longrightarrow a$, $y_n \longrightarrow b$. *Then*

a) $x_n \pm y_n \longrightarrow a \pm b$;

b) $x_n y_n \longrightarrow a b$;

c) If $b \neq 0$, *then* $x_n / y_n \longrightarrow a / b$.

Proof. We only give detailed proof for b). Other cases are left as exercises.

To show $x_n y_n \longrightarrow a b$, all we need is for any given $\varepsilon > 0$, find $N \in \mathbb{N}$ such that for all $n > N$, $|x_n y_n - a b| < \varepsilon$. First assume that $a, b \neq 0$.

Let $N_1 \in \mathbb{N}$ be such that $|y_n - b| < |b|$. Now for any given $\varepsilon > 0$, let N_2 be such that $|x_n - a| < \frac{\varepsilon}{4 |b|}$, and N_3 be such that $|y_n - b| < \frac{\varepsilon}{2 |a|}$.

Take $N = \max \{N_1, N_2, N_3\}$. For any $n > N$, we have

$$\begin{aligned} |x_n y_n - a b| &= |(x_n - a) y_n + a (y_n - b)| \\ &\leqslant |(x_n - a)| |y_n| + |a| |y_n - b| \\ &< \frac{\varepsilon}{4 |b|} 2 |b| + |a| \frac{\varepsilon}{2 |a|} \\ &= \varepsilon. \end{aligned}$$

When either a or b is (or both are) 0, all we need to show is $x_n y_n \longrightarrow 0$. The method is the same and the argument is simpler so omitted. □

Exercise 7. Prove a).

Exercise 8. Find the mistake in the following proof of c):

Let $\lim_{n \longrightarrow \infty} \frac{x_n}{y_n} = c$. Then

$$a = \lim_{n \longrightarrow \infty} x_n = \lim_{n \longrightarrow \infty} \left(\frac{x_n}{y_n} \right) \cdot y_n = \left(\lim_{n \longrightarrow \infty} \frac{x_n}{y_n} \right) \left(\lim_{n \longrightarrow \infty} y_n \right) = c b. \tag{15}$$

So $c = a / b$.

Then give a correct proof of c).

The above proposition can be readily extended to cover the cases of $\pm\infty$ limits. Recall the following rules:

$$\begin{aligned} x + \infty = \infty, &\qquad x - \infty = -\infty, &\qquad x \in \mathbb{R} &\qquad (16) \\ x \cdot \infty = \infty, &\qquad x \cdot (-\infty) = -\infty, &\qquad x > 0 &\qquad (17) \\ x \cdot \infty = -\infty, &\qquad x \cdot (-\infty) = \infty, &\qquad x < 0 &\qquad (18) \\ \infty + \infty = \infty, &\qquad -\infty - \infty = -\infty & &\qquad (19) \\ \infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty &\qquad \infty \cdot (-\infty) = (-\infty) \cdot (\infty) = -\infty & &\qquad (20) \end{aligned}$$

Proposition 8. (Extension of Proposition 7) *Let* $x_n \longrightarrow a$, $y_n \longrightarrow b$ *where* a, b *are extended real numbers. Then*

a) $x_n \pm y_n \longrightarrow a \pm b$;

b) $x_n y_n \longrightarrow a b$;

c) If $b \neq 0$, *then* $x_n / y_n \longrightarrow a / b$.

as long as the right hand sides are well-defined in the arithmetics of extended real numbers.

Proof. We only prove the product case and leave other cases as exercises.

When both $a, b \in \mathbb{R}$, the proof has been done in Proposition 7. Here we need to study the two new cases: $a, b = \pm\infty$ and $a \in \mathbb{R}, b = \pm\infty$ or $a = \pm\infty, b \in \mathbb{R}$.

- Case 1. We only prove for $a = b = \infty$. (The other three sub-cases can be proved similarly.) As $\infty \cdot \infty = \infty$, we need to show that for every $M \in \mathbb{R}$, there is $N \in \mathbb{N}$ such that for all $n > N$, $x_n y_n > M$.

  For every given $M \in \mathbb{R}$, since $x_n \longrightarrow \infty$, there is $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $x_n > \sqrt{|M|}$; Similarly there is $N_2 \in \mathbb{N}$ such that for all $n > N_2$, $y_n > \sqrt{|M|}$. Now take $N = \max\{N_1, N_2\}$. For any $n > N$, we have $x_n y_n > \sqrt{|M|}\sqrt{|M|} = |M| \geqslant M$.

- Case 2. We only prove for $a > 0$, $b = \infty$ (The other seven sub-cases can be proved similarly). We need to show that for every $M \in \mathbb{R}$, there is $N \in \mathbb{N}$ such that for all $n > N$, $x_n y_n > M$.

  Since $x_n \longrightarrow a > 0$, there is $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $x_n > a/2$. On the other hand, as $y_n \longrightarrow \infty$, there is $N_2 \in \mathbb{N}$ such that for all $n > N_2$, $y_n > 2|M|/a$. Now take $N = \max\{N_1, N_2\}$, for every $n > N$ we have $x_n y_n > (a/2)(2|M|/a) = |M| \geqslant M$. □

Exercise 9. Explain why in the following cases the limt is not determined by a, b only

$$\infty - \infty;\ 0 \cdot \infty;\ \infty/\infty \tag{21}$$

through examples for each cases.

Remark 9. These types of limit can be calculated using L'Hospitale's rules.

Theorem 10. (Comparison of limits) *Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences with limits $a, b \in \mathbb{R}_{\text{ext}}$. If there is $N_0 \in \mathbb{N}$ such that $x_n \leqslant y_n$ for all $n \geqslant N_0$, then*

$$\lim_{n\longrightarrow\infty} x_n \leqslant \lim_{n\longrightarrow\infty} y_n. \tag{22}$$

Proof.

- The case $a, b \in \mathbb{R}$.

  Let $x \longrightarrow a$, y — b. We prove by contradiction. Assume $a > b$. Set $\varepsilon = \frac{a-b}{2}$. Then there is $N_1 \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n > N_1$, while $N_2 \in \mathbb{N}$ such that $|y_n - b| < \varepsilon$ for all $n > N_2$. Take $n > \max\{N_1, N_2\}$. Then we have

  $$x_n \geqslant a - |x_n - a| > a - \varepsilon = \frac{a+b}{2}; \tag{23}$$

  $$y_n \leqslant b + |y_n - b| < b + \varepsilon = \frac{a+b}{2}. \tag{24}$$

  But this combined with $x_n \leqslant y_n$ gives $\frac{a+b}{2} < \frac{a+b}{2}$, contradiction.

- The case $a = -\infty$. As $-\infty \leqslant b$ for every extended real number b, the proof ends.
- The case $a = +\infty$. We need to show that $b = -\infty$ or $b \in \mathbb{R}$ would lead to contradiction.
  - If $b = -\infty$. By definition we have

    $$\exists N_1 \in \mathbb{N}\ \forall n > N_1 \qquad y_n < -1; \tag{25}$$

    On the other hand, $\lim x_n = a = +\infty$ leads to

    $$\exists N_2 \in \mathbb{N}\ \forall n > N_2 \qquad x_n > 1. \tag{26}$$

    Taking any $n > \max\{N_1, N_2\}$, we have

    $$x_n > 1 > -1 > y_n \tag{27}$$

    contradiction.
  - $b \in \mathbb{R}$. Left as exercise.
- The cases $b = \pm\infty$ can be proved similarly and are left as exercises.

□

Exercise 10. Prove the cases $a=+\infty, b\in\mathbb{R}$ and $b=\pm\infty$.

Exercise 11. Is the following claim true or false? Justify your answer.

> Suppose $\{x_n\},\{y_n\}$ are convergent sequences. If there is $N_0\in\mathbb{N}$ such that $x_n<y_n$ for all $n\geqslant N_0$, then $\lim_{n\longrightarrow\infty}x_n<\lim_{n\longrightarrow\infty}y_n$.

Lemma 11. *A sequence can have at most one limit.*

Proof. We prove by contradiction. Assume $x_n\longrightarrow a$ and $x_n\longrightarrow b$ with $a\neq b$. Without loss of generality we assume $b>a$.

Take $\varepsilon=\frac{b-a}{2}$. There is $N_1\in\mathbb{N}$ such that $|x_n-a|<\varepsilon$ for all $n>N_1$; There is $N_2\in\mathbb{N}$ such that $|x_n-b|<\varepsilon$. Now take $N=\max\{N_1,N_2\}$. Now take any $n>N$, we have $n>N_1, n>N_2$ and therefore

$$|x_n-a|<\varepsilon, |x_n-b|<\varepsilon \tag{28}$$

which combined to give

$$b-a=|b-a|\leqslant|b-x_n|+|x_n-a|=|x_n-b|+|x_n-a|<2\,\varepsilon=b-a. \tag{29}$$

Contradiction! □

An important method of guessing/proving limit is the following theorem.

Theorem 12. (Squeeze Theorem) *Let $a\in\mathbb{R}_{\text{ext}}$, $x_n\rightarrow a$ and $y_n\longrightarrow a$. Let $\{w_n\}$ be a sequence. Assume that there is $N_0\in\mathbb{N}$ such that for all $n\geqslant N_0$,*

$$x_n\leqslant w_n\leqslant y_n, \tag{30}$$

then

a) w_n converges.

b) $w_n\longrightarrow a$.

Remark 13. Of course proving $w_n\longrightarrow a$ suffices. However we would like to emphasize that, the significance of this theorem is that the relation (29) "forces" w_n to converge.

Proof. We prove the case $a\in\mathbb{R}$. For any given $\varepsilon>0$, let $N_1\in\mathbb{N}$ be such that $|x_n-a|<\varepsilon$ for all $n>N_1$; Let $N_2\in\mathbb{N}$ be such that $|y_n-a|<\varepsilon$ for all $n>N_2$. Now let $N=\max\{N_1,N_2\}$. For all $n>N$, we have

$$|w_n-a|\leqslant\max\{|x_n-a|,|y_n-a|\}<\varepsilon. \tag{31}$$

Thus ends the proof. □

Exercise 12. Prove the cases $a=\infty$ and $a=-\infty$. Do we really need both $\{x_n\},\{y_n\}$ in these two cases? Justify your answers.

Corollary 14. *Let $\{w_n\}$ be a sequence. If there is another sequence $x_n\longrightarrow 0$ such that $|w_n|\leqslant x_n$, then $w_n\longrightarrow 0$. In particular, if $|w_n|\longrightarrow 0$, then $w_n\longrightarrow 0$.*

Exercise 13. Prove the above corollary.

Example 15. Find $\lim_{n\longrightarrow\infty}2^{-n}\sin(n^8)$.

We simply apply the above corollary: $2^{-n}\longrightarrow 0$,

$$|2^{-n}\sin(n^8)|\leqslant 2^{-n} \tag{32}$$

therefore the limit is also 0.

1.3. Cauchy sequence.

Definition 16. (Cauchy sequence) *A sequence of real numbers $\{x_n\}$ is called a* Cauchy sequence *if and only if the following holds:*

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall m, n > N, \qquad |x_m - x_n| < \varepsilon. \tag{33}$$

Often one simply says "$\{x_n\}$ is Cauchy".

Exercise 14. Obtain the working negation of "$\{x_n\}$ is Cauchy".

Exercise 15. Why is $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that $\forall n > N$, $|a_{n+1} - a_n| < \varepsilon$ not enough?

Recall that, to prove convergence for $\{x_n\}$ with formulas for x_n given, we need to do two things:

1. Guess the limit a.
2. Prove that indeed $x_n \longrightarrow a$ using either definition or properties (see Section ? below) or both.

On the other hand, there are many situations where we do not have explicit formulas for x_n and thus cannot guess what the limit is. The following is the most useful result in those situations.

Theorem 17. (Relation to Cauchy sequences) *Let $\{x_n\}$ be a sequence. Then $\{x_n\}$ converges to some real number $a \in \mathbb{R}$ if and only if it is a Cauchy sequence.*

Proof. It's "if and only if", so we need to show the "only if":

$$x_n \longrightarrow a \text{ for some } a \in \mathbb{R} \Longrightarrow \{x_n\} \text{ is Cauchy} \tag{34}$$

and the "if":

$$\{x_n\} \text{ is Cauchy} \Longrightarrow x_n \longrightarrow a \text{ for some } a \in \mathbb{R}. \tag{35}$$

- "Only if". Given any $\varepsilon > 0$, we need to find $N \in \mathbb{N}$ such that for all $n, m > N$, $|x_n - x_m| < \varepsilon$. We proceed as follows. For the given $\varepsilon > 0$, since $x_n \longrightarrow a$, there is $N \in \mathbb{N}$ such that

$$|x_n - a| < \varepsilon/2 \tag{36}$$

  for all $n > N$. Now for any $n, m > N$, we have

$$|x_n - a| < \varepsilon/2, \qquad |x_m - a| < \varepsilon/2 \tag{37}$$

  which combined to give

$$|x_n - x_m| = |(x_n - a) + (a - x_m)| \leqslant |x_n - a| + |a - x_m| = |x_n - a| + |x_m - a| < \varepsilon. \tag{38}$$

- "If". For this part we need to use the least upper bound property of $\mathbb{R}$. Intuitively, $\{x_n\}$ is Cauchy means they cluster around something, but to show that "something" is a real number, we need the property that $\mathbb{R}$ has no "holes".

Let the set $A = \{a \in \mathbb{R}: \exists N \in \mathbb{N},$ such that $a < x_n$ for all $n > N\}$. We first show that A has an upper bound. Since $\{x_n\}$ is Cauchy, there is $N_1 \in \mathbb{N}$ such that $|x_n - x_m| < 1$ for all $n, m > N_1$. In particular we have

$$x_n < x_{N_1+1} + 1 \tag{39}$$

for all $n > N_1$. Now take

$$b = \max\{x_1 + 1, x_2 + 1, \ldots, x_{N_1} + 1, x_{N_1+1} + 1\} \tag{40}$$

we have $b > x_n$ for all $n \in \mathbb{N}$ and clearly $b > a$ for every $a \in A$. Using the same method one can show that A is not empty.

Thanks to the least upper bound property, we have $b_{\min} \in \mathbb{R}$ such that $b_{\min} \geqslant a$ for every $a \in A$ while for every $b < b_{\min}$ there is $a \in A$ with $a > b$. We prove the $x_n \longrightarrow b_{\min}$.

Given any $\varepsilon > 0$. We know that there is $a \in A$ such that $a > b_{\min} - \varepsilon$. Therefore there is $N_1 \in \mathbb{N}$ such that $b_{\min} - \varepsilon < x_n$ for all $n > N_1$;

Now we show the existence of $N_2 \in \mathbb{N}$ such that $b_{\min} + \varepsilon > x_n$ for all $n > N_2$. We prove by contradiction. Assume that for every $N \in \mathbb{N}$, there is $n > N$ such that $x_n \geqslant b_{\min} + \varepsilon$. Since $\{x_n\}$ is a Cauchy sequence, there is $N_3 \in \mathbb{N}$ such that for all $n, m > N_3$, $|x_n - x_m| < \varepsilon/2$. For this N_3 we can find $l > N_3$ with $x_l \geqslant b_{\min} + \varepsilon$. Consequently, for all $n > N_3$, we have

$$x_n \geqslant x_l - |x_n - x_l| > b_{\min} + \varepsilon/2. \tag{41}$$

But this means $b_{\min} + \varepsilon/2 \in A$, contradicting $b_{\min} \geqslant a$ for every $a \in A$.

Finally take $N = \max\{N_1, N_2\}$. For every $n > N$, we have

$$b_{\min} - \varepsilon < x_n < b_{\min} + \varepsilon \Longrightarrow |x_n - b_{\min}| < \varepsilon. \tag{42}$$

Therefore $x_n \longrightarrow b_{\min}$ and the proof for "if" ends. □

Exercise 16. Can this theorem be extended to the cases $a = \pm\infty$? Why?

Example 18. Let $x_0 > 2$. Define x_n iteratively by

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2\,x_n}. \tag{43}$$

Prove that $x_n \longrightarrow \sqrt{2}$.

Proof. The idea is to show that it is Cauchy. Once this is done, we can take limits of both sides to reach (denote the limit by a):

$$a = a - \frac{a^2 - 2}{2\,a} \Longrightarrow a^2 = 2 \Longrightarrow a = \pm\sqrt{2}. \tag{44}$$

From this we see that we need to prove the following:

- First show that $x_n > 0$. We use mathematical induction. Let $P(n)$ denote the statement "$x_n > 0$". Mathematical induction consists of two steps:

  1. $P(0)$ is true.[2] We have $x_0 > 2 > 0$.

  2. If $P(m)$ is true then $P(m+1)$ is true. We have

  $$x_{m+1} = x_m - \frac{x_m^2 - 2}{2\, x_m} = \frac{x_m^2 + 2}{2\, x_m}. \tag{45}$$

  Therefore if $x_m > 0$, $x_{m+1} > 0$.

- Since a ratio is involved, we need to show $x_n \neq 0$ and $a \neq 0$. We do this through showing $x_n^2 \geqslant 2$ for all $n \in \mathbb{N}$. This can be done directly as follows:

$$x_n^2 - 2 = \left(x_{n-1} - \frac{x_{n-1}^2 - 2}{2\, x_{n-1}} \right)^2 - 2 = \left(\frac{x_{n-1}^2 + 2}{2\, x_{n-1}} \right)^2 - 2 = \left(\frac{x_{n-1}^2 - 2}{2\, x_{n-1}} \right)^2 \geqslant 0. \tag{46}$$

- x_n is Cauchy. The basic idea is to show that $|x_n - x_{n+1}| \leqslant M\, r^n$ for some $0 < r < 1$ and some $M > 0$ (Why this is enough is left as exercise). Taking difference of

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2\, x_n} = \frac{x_n}{2} + \frac{1}{x_n} \text{ and } x_n = x_{n-1} - \frac{x_{n-1}^2 - 2}{2\, x_{n-1}} = \frac{x_{n-1}}{2} + \frac{1}{x_{n-1}} \tag{47}$$

  we have

$$x_{n+1} - x_n = \left[\frac{1}{2} - \frac{1}{x_n\, x_{n-1}} \right] (x_n - x_{n-1}). \tag{48}$$

  Therefore

$$|x_{n+1} - x_n| = \left| \frac{1}{2} - \frac{1}{x_n\, x_{n-1}} \right| |x_n - x_{n-1}|. \tag{49}$$

  As $x_n > \sqrt{2}$ we have $x_n\, x_{n-1} \geqslant 2$ so $\left| \frac{1}{2} - \frac{1}{x_n\, x_{n-1}} \right| \leqslant \frac{1}{2}$. This leads to

$$|x_{n+1} - x_n| \leqslant \frac{1}{2} |x_n - x_{n-1}| \leqslant \frac{1}{2^2} |x_{n-1} - x_{n-2}| \leqslant \cdots \leqslant \left(\frac{1}{2} \right)^n |x_1 - x_0|. \tag{50}$$

Since x_n is Cauchy, there is $a \in \mathbb{R}$ such that $x_n \longrightarrow a$. By (44) we have $a = \pm\sqrt{2}$. Since $x_n > 0$, we conclude $a = \sqrt{2}$ (using Comparison Theorem 7). □

Remark 19. The above is a very effective way to compute $\sqrt{2}$.

2. Because we start from x_0.

1.4. Understanding convergence.

In this section we understand what convergence means. Or more precisely, we understand what happens when a sequence is **not** convergent. Turns out, there are only two situations where a sequence does not converge:

1. The sequence is unbounded.
2. The sequence is oscillating and the amplitude does not tend to 0.

Definition 20. (Boundedness) *A sequence $\{x_n\}$ is said to be*

- *bounded above if there is a number $M \in \mathbb{R}$ such that $x_n \leqslant M$ for all n,*
- *bounded below if there is a number $M \in \mathbb{R}$ such that $x_n \geqslant M$ for all n,*
- *bounded if there is a number $M \in \mathbb{R}$ such that $|x_n| \leqslant M$ for all n.*

Exercise 17. Obtain the working negation of "$\{x_n\}$ is bounded".

Lemma 21. (Relation between boundedness) *$\{x_n\}$ is bounded if and only if it is bounded above and below.*

Proof.

- "If". If $\{x_n\}$ is bounded above and below, then there are $M_1, M_2 \in \mathbb{R}$ such that $M_1 \leqslant x_n \leqslant M_2$ for all $n \in \mathbb{N}$. Now take $M = \max\{|M_1|, |M_2|\}$. We have

$$-M \leqslant -|M_1| \leqslant M_1 \leqslant x_n \leqslant M_2 \leqslant |M_2| \leqslant M \tag{51}$$

  for all $n \in \mathbb{N}$.

- "Only if". If $\{x_n\}$ is bounded, then there is $M \in \mathbb{R}$ such that $-M \leqslant x_n \leqslant M$. Thus $\{x_n\}$ is bounded both above and below. □

"Oscillating" is quantified through "subsequences" converging to different limits.

Definition 22. (Subsequence) *A* subsequence *of a sequence $\{x_n\}_{n \in \mathbb{N}}$ is a sequence of the form $\{x_{n_k}\}_{k \in \mathbb{N}}$, where each $n_k \in \mathbb{N}$ with $n_1 < n_2 < n_3 < \cdots$.*

Remark 23. It is important to understand that the "n" in the subsequence $\{x_{n_k}\}$ is just a notation, it does not take any value. It is k that takes values $1, 2, 3, \ldots$.

Remark 24. Recall that a real sequence is a function $f: \mathbb{N} \mapsto \mathbb{R}$. Then its subsequence is the same function restricted to a subset $E \subset \mathbb{R}$.

Exercise 18. A key fact for subsequences is

$$n_k \geqslant k \qquad \text{for all } k \in \mathbb{N}. \tag{52}$$

Prove it. (Hint: Use mathematical induction.)

Exercise 19. Another way to understand subsequences is to view them as composite functions. Figure this out.

It is clear that a subsequence of a subsequence is a subsequence of the original sequence.

1.4.1. Subsequences and convergence.

Lemma 25. *Let $a \in \mathbb{R}_{\text{ext}}$. We have the following relations between convergence of $\{x_n\}$ and its subsequences.*

a) *If $x_n \longrightarrow a$, then every of its subsequences also converge to a.*

b) *If all its subsequences converge to the same $a \in \mathbb{R}_{\text{ext}}$, then $x_n \longrightarrow a$.*

Proof. We prove a) for $a = -\infty$ and b) for $a \in \mathbb{R}$ and leave other cases as exercises.

a) We have $x_n \longrightarrow -\infty$ and need to prove $x_{n_k} \longrightarrow -\infty$ for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$. More precisely, our hypothesis is

$$\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n > N \qquad x_n < M \tag{53}$$

and the conclusion to be proved is

$$\forall M \in \mathbb{R}\ \exists K \in \mathbb{N}\ \forall k > K \qquad x_{n_k} < M. \tag{54}$$

Given an arbitrary $M \in \mathbb{R}$. Let $N \in \mathbb{N}$ be such that $\forall n > N$, $x_n < M$. Now take $K = N$. Then for every $k > K$, we have $n_k \geqslant k > K = N$ which gives

$$x_{n_k} < M. \tag{55}$$

b) We provce by contradiction. Assume $\{x_n\}$ does not converge to a. All we need to do is to find one subsequence which does not converge to a[3]. As $\{x_n\}$ does not converge to a, we have

$$\exists \varepsilon_0 > 0\ \forall N \in \mathbb{N}\ \exists n > N, \qquad |x_n - a| > \varepsilon_0. \tag{56}$$

Now first take $N = 1$. There is $n_1 > 1$ such that $|x_{n_1} - a| > \varepsilon_0$. Next take $N = n_1$. There is $n_2 > N = n_1$ such that $|x_{n_2} - a| > \varepsilon_0$. Next take $N = n_2$ and do the same and so on. This way we obtain a subsequence $\{x_{n_k}\}$ satisfying $|x_{n_k} - a| > \varepsilon_0$ for all $k \in \mathbb{N}$. This subsequence does not converge to a. □

Exercise 20. Prove the above lemma for all other cases.

1.4.2. Bolzano-Weierstrass.

Theorem 26. (Bolzano-Weierstrass) *Let $\{x_n\}$ be bounded a sequence of real numbers. Then there is a converging subsequence.*

Remark 27. The theorem not only tell us something about simple cases, such as $(-1)^n$ for which we can easily get a convergent subsequence, it also tell us sequences like $\{\sin n\}$ also has convergent subsequence(s).[4]

Monotone sequences.

3. Whether it converges to something else or does not converge at all does not matter here.

4. In fact, for every $-1 \leqslant a \leqslant 1$, there is a subsequence of $\{\sin n\}$ convergent to a.

To prove the Bolzano-Weierstrass theorem, we first study monotone sequences.

Definition 28. *A sequence $\{x_n\}$ is increasing if $x_{n+1} \geqslant x_n$ for all n (strictly increasing of $x_{n+1} > x_n$ for all $n \in \mathbb{N}$); A sequence is decreasing if $x_{n+1} \leqslant x_n$ for all n (strictly decreasing if $x_{n+1} < x_n$ for all $n \in \mathbb{N}$); A sequence is monotone if it is either increasing or decreasing.*

Convergence results when monotonicity meets boundedness.

Lemma 29. *Let $\{x_n\}$ be monotone increasing (decreasing) and bounded above (below). Then $x_n \longrightarrow a$ for some $a \in \mathbb{R}$.*

Proof. Since $\{x_n\}$ is bounded above. There is $M \in \mathbb{R}$ such that $x_n \leqslant M$ for all $n \in \mathbb{N}$.

We now show that x_n is Cauchy through proof by contradiction. Assume x_n is not Cauchy, then there is $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$, there are $n, m > N$ with $|x_n - x_m| > \varepsilon_0$.

Now take $N = 1$. We find $n_2 > n_1 > 1$ with $|x_{n_2} - x_{n_1}| > \varepsilon_0$. Next take $N = n_2$, we have $n_4 > n_3 > n_2$ with $|x_{n_4} - x_{n_3}| > \varepsilon_0$. Doing this repeatedly, we obtain a (still increasing) subsequence x_{n_k} such that $|x_{n_{2k}} - x_{n_{2k-1}}| > \varepsilon_0$. Since this subsequence is still increasing,

$$x_{n_{2k}} \geqslant (x_{n_{2k}} - x_{n_{2k-1}}) + (x_{n_{2k-2}} - x_{n_{2k-3}}) + \cdots + (x_{n_2} - x_{n_1}) + x_{n_1} > k\,\varepsilon_0 + x_{n_1}. \tag{57}$$

Taking $k > (M - x_{n_1})/\varepsilon_0$ we reach $x_{n_{2k}} > M$. Contradiction. □

Remark 30. An alternative proof is as follows: Let $A = \{a \in \mathbb{R}: a < x_n \text{ for some } n \in \mathbb{N}\}$. We can show that A has an upper bound and is nonempty. Applying the least upper bound property of $\mathbb{R}$, there is $b_{\min} \in \mathbb{R}$ such that $b_{\min} \geqslant a$ for all $a \in A$, and for every $b < b_{\min}$, there is $a \in A$ satisfying $a > b$. Then prove $x_n \longrightarrow b_{\min}$. This approach is left as exercise.

Proof of Bolzano-Weierstrass.

All we need to do is to find a monotone subsequence from $\{x_n\}$.

Proof. **(of Bolzano-Weierstrass)** Consider the following subsequences of $\{x_n\}$: x_m, x_{m+1}, $x_{m+2}, \ldots$. There are only two cases:

- For every such sequence, there is a smallest element: $x_{n_1} = x_1$,

  $$x_{n_k} := \min\{x_{n_{k-1}+1}, x_{n_{k-1}+2}, \ldots\}. \tag{58}$$

  In this case we know that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and it is increasing and bounded above. Applying Lemma 29 to this subsequence we see that it converges to some $a \in \mathbb{R}$.

- There is one m, such that there is no smallest element in $\{x_{n_m+1}, x_{n_m+2}, \ldots\}$. Set $\tilde{x}_{n_1} = x_{n_m+1}$. Set n_2 to be the smallest number that $\tilde{x}_{n_2} < \tilde{x}_{n_1}$, that is $\tilde{x}_i \geqslant \tilde{x}_{n_1}$ for all $n_1 \leqslant i < n_2$. This can be done because there is no smallest number. Set n_3 to be the smallest number that $\tilde{x}_{n_3} < \tilde{x}_{n_2}$, and so on. This way we obtained a decreasing subsequence which is bounded below. Applying Lemma 29 to this subsequence we see that it converges to some $a \in \mathbb{R}$. □

1.4.3. Classification of sequences.

Lemma 31. *We have the following relation between convergence and boundedness.*

a) If $x_n \longrightarrow a \in \mathbb{R}$, then $\{x_n\}$ is bounded.

b) If $\{x_n\}$ is bounded, then there is a subsequence $\{x_{n_k}\}$ that converges to some $a \in \mathbb{R}$.

Proof.

a) Take $\varepsilon = 1$. Since $x_n \longrightarrow a$, there is $N \in \mathbb{N}$ such that for all $n > N$, we have $|x_n - a| < \varepsilon$. Thus for all $n > N$, we have

$$|x_n| \leqslant |a| + |x_n - a| < |a| + 1. \tag{59}$$

Now set

$$M = \max\{|x_1|, \ldots, |x_N|, |a| + 1\}. \tag{60}$$

We have $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. Thus ends the proof.

b) This follows immediately from Bolzano-Weierstrass. □

Exercise 21. Spot the mistake in the following proof of part a):

Take $\varepsilon = |a|$. Since $x_n \longrightarrow a$, ... (the remaining the same as that in the above proof)

Now we almost have proved the following theorem.

Theorem 32. (Classification of sequences) *Let $\{x_n\}$ be a sequence of real numbers. Then one of the following is true:*

a) x_n converges to $a \in \mathbb{R}$;

b) $\{x_n\}$ is not bounded.

c) There are two convergent subsequences with different limits.

Remark 33. a) and b),c) are clearly mutually exclusive. On the other hand both b) c) may be true for a single sequence. For example let $a_n = \begin{cases} 1 & n = 4k+1 \\ n & n = 4k+2 \\ -1 & n = 4k+3 \\ -n & n = 4k+4 \end{cases}$ for $k \in \mathbb{N}$.

Proof. (of the Theorem) It is clear that a) and b) are mutually exclusive. The only thing left to show is that if $\{x_n\}$ is bounded and does not converge, then it has two converging subsequences with different limits.

Since $\{x_n\}$ is bounded, by the Bolzano-Weierstrass theorem there is a converging subsequence. We denote its limit by a. Since $x_n \not\longrightarrow a$, by definition there must be $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$ there is $n > N$ with $|x_n - a| > \varepsilon_0$. From this we can get a subsequence satisfying $|x_{n_k} - a| > \varepsilon_0$ as follows:

i. Take $N = 1$. There is $|x_{n_1} - a| > \varepsilon_0$;

ii. Take $N = n_1$. Find $|x_{n_2} - a| > \varepsilon_0$;

iii. Take $N = n_2$, ...

Now apply the Bolzano-Weierstrass theorem to this subsequence. We see that there is a converging subsequence (to this subsequence), called $\left\{x_{n_{k_l}}\right\}$ ($l = 1, 2, 3, \ldots$). Let $x_{n_{k_l}} \longrightarrow a'$ as $l \longrightarrow \infty$. We show that $a' \neq a$. Assume the contrary. Then $x_{n_{k_l}} \longrightarrow a$. By definition there is $N \in \mathbb{N}$ such that $\left|x_{n_{k_l}} - a\right| < \varepsilon_0$ for all $l > N$. But this contradicts the fact that $x_{n_{k_l}}$ are chosen from x_{n_k} which satisfies $|x_{n_k} - a| > \varepsilon_0$ for all $k \in \mathbb{N}$. □

Exercise 22. Prove the following:

If $\{x_n\}$ is unbounded, then one of the following is true:

a) $x_n \longrightarrow +\infty$;

b) $x_n \longrightarrow -\infty$;

c) There are two subsequences, one converges to $+\infty$ and one to $-\infty$.

1.4.4. Liminf and Limsup.

Definition 34. *Let $\{x_n\}$ be a real sequence. The* limit supreme *of $\{x_n\}$ is the extended real number*

$$\limsup_{n \longrightarrow \infty} x_n := \lim_{n \longrightarrow \infty} \left(\sup_{k \geqslant n} x_k \right), \tag{61}$$

and the limit infimum *of $\{x_n\}$ is the extended real number*

$$\liminf_{n \longrightarrow \infty} x_n := \lim_{n \longrightarrow \infty} \left(\inf_{k \geqslant n} x_k \right). \tag{62}$$

Remark 35. Since $\sup_{k \geqslant n} x_k$ is decreasing and $\inf_{k \geqslant n} x_k$ is increasing, limsup and liminf always exist (in contrast to limits).

Remark 36. One way to understand limsup, liminf is the following.

Let $\{x_n\}$ be a sequence. Then we create a new sequence $\{y_n\}$ as follows.

$$y_1 := \sup \{x_1, x_2, \ldots\}; \tag{63}$$

$$y_2 := \sup \{x_2, x_3, \ldots\}; \tag{64}$$

$$\vdots$$

$$y_n := \sup \{x_n, x_{n+1}, \ldots\}. \tag{65}$$

$$\vdots$$

Now as y_n is decreasing, its limit exists (either in $\mathbb{R}$ or is ∞). We call this limit the limit supreme of $\{x_n\}$ and denote it by $\limsup_{n \rightarrow \infty} x_n$.

For example, if $x_n = (-1)^n$, then

$$y_1 := \sup \{-1, 1, -1, \ldots\} = 1; \tag{66}$$

$$y_2 := \sup \{1, -1, 1, \ldots\} = 1; \tag{67}$$

$$\vdots$$

$$y_n := \sup \{(-1)^n, (-1)^{n+1}, (-1)^{n+2}, \ldots\} = 1. \tag{68}$$

$$\vdots$$

So in fact $y_n = 1$ for all n. We have $\lim_{n \longrightarrow \infty} y_n = 1$ which means $\limsup_{n \rightarrow \infty} x_n = 1$.

Remark 37. Note that it is possible that $\limsup_{n \longrightarrow \infty} x_n = -\infty$ and $\liminf_{n \longrightarrow \infty} = \infty$.

Lemma 38. *Let $\{x_n\}$ be a real sequence. Then*

$$\limsup_{n \longrightarrow \infty} x_n \geqslant \liminf_{n \longrightarrow \infty} x_n \tag{69}$$

Proof. Follows from Comparison Theorem. Left as exercise. □

Exercise 23. Prove the above lemma.

Theorem 39. (Relation to limits and subsequences) *Let* $\{x_n\}$ *be a real sequence. Then*

a) $x_n \longrightarrow a$ *if and only if* $\mathrm{limsup}_{n\longrightarrow\infty} x_n = \mathrm{liminf}_{n\longrightarrow\infty} x_n = a$.[5]

b) There are two subsequences converging to $\mathrm{limsup}_{n\longrightarrow\infty} x_n$ *and* $\mathrm{liminf}_{n\longrightarrow\infty} x_n$*, respectively.*

c) If $\{x_{n_k}\}$ *is a convergent subsequence, then* $\mathrm{liminf}_{n\longrightarrow\infty} x_n \leqslant \lim_{k\longrightarrow\infty} x_{n_k} \leqslant \mathrm{limsup}_{n\longrightarrow\infty} x_n$.

Proof.

a) We only prove the case $a \in \mathbb{R}$ and left $a = \pm\infty$ as exercise.

- "if". Assume $\mathrm{limsup}_{n\longrightarrow\infty} x_n = \mathrm{liminf}_{n\longrightarrow\infty} x_n = a$. Then as

$$\sup_{k\geqslant n} x_k \geqslant x_n \geqslant \inf_{k\geqslant n} x_k, \tag{70}$$

  application of Squeeze Theorem gives the convergence of x_n as well as $\lim_{n\longrightarrow\infty} x_n = a$.

- "only if". For any $\varepsilon > 0$, since $x_n \longrightarrow a$, there is $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$. This implies $\sup_{n>N} x_n < a + \varepsilon$ and $\inf_{n>N} x_n > a - \varepsilon$. Therefore when $n > N$,

$$a - \varepsilon < \inf_{k\geqslant n} x_n \leqslant \sup_{k\geqslant n} x_n < a + \varepsilon. \tag{71}$$

  Comparison Theorem now gives

$$a - \varepsilon \leqslant \liminf_{n\longrightarrow\infty} x_n \leqslant \limsup_{n\longrightarrow\infty} x_n \leqslant a + \varepsilon. \tag{72}$$

  As this holds for all $\varepsilon > 0$, we must have $\mathrm{limsup}_{n\longrightarrow\infty} x_n = \mathrm{liminf}_{n\longrightarrow\infty} x_n = a$.

b) We only show the existence of $x_{n_k} \longrightarrow \mathrm{limsup}_{n\longrightarrow\infty} x_n$. Thanks to Theorem ?, there is $n_1 \in \mathbb{N}$ such that

$$\sup_{k\geqslant 1} x_k - 1 \leqslant x_{n_1} \leqslant \sup_{k\geqslant 1} x_k; \tag{73}$$

Now apply Theorem ? to $\{x_{n_1+1}, \ldots\}$, we obtain $n_2 \in \mathbb{N}$ such that

$$\sup_{k\geqslant n_2+1} x_k - \frac{1}{2} \leqslant x_{n_2} \leqslant \sup_{k\geqslant n_2+1} x_k. \tag{74}$$

This way we obtain a subsequence satisfying

$$\sup_{k\geqslant n_l+1} x_k - \frac{1}{l} \leqslant x_{n_l} \leqslant \sup_{k\geqslant n_l+1} x_k. \tag{75}$$

We take limit of both sides. As $\{\sup_{k\geqslant n_l+1} x_k\}$ is a subsequence of $\{\sup_{k\geqslant n} x_k\}$, it converges to the same limit a. Using the fact that $1/l \longrightarrow 0$ as $l \longrightarrow \infty$, we apply Squeeze theorem to conclude $x_{n_l} \longrightarrow a$.

c) Let $\{x_{n_k}\}$ be the subsequence. Then we have

$$\inf_{l\geqslant n_k} x_l \leqslant x_{n_k} \leqslant \sup_{l\geqslant n_k} x_l. \tag{76}$$

Taking limit of all three and applying Comparison Theorem we reach the conclusion. □

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5. *We do not say "*$\{x_n\}$ *converges" to avoid dealing with* $a = \pm\infty$ *separately.*

Remark 40. More discussions about liminf and limsup.

- Definition:

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{k\geqslant n} x_k = \lim_{n\longrightarrow\infty} \sup \{x_n, x_{n+1}, \ldots\}. \tag{77}$$

$$\liminf_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} \inf_{k\geqslant n} x_k = \lim_{n\longrightarrow\infty} \inf \{x_n, x_{n+1}, \ldots\}. \tag{78}$$

  In other words, we define a new sequence by

$$\begin{aligned} y_1 &= \sup \{x_1, x_2, x_3, \ldots\} & (79)\\ y_2 &= \sup \{x_2, x_3, x_4, \ldots\} & (80)\\ &\vdots \\ y_n &= \sup \{x_n, x_{n+1}, x_{n+2}, \ldots\} & (81)\\ &\vdots \end{aligned}$$

  and then define

$$\limsup_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} y_n. \tag{82}$$

- Understanding:

$$\limsup_{n\longrightarrow\infty} x_n = \max \{a \in \mathbb{R}: \exists \text{subsequence } x_{n_k} \longrightarrow a\}; \tag{83}$$

$$\liminf_{n\longrightarrow\infty} x_n = \min \{a \in \mathbb{R}: \exists \text{subsequence } x_{n_k} \longrightarrow a\}. \tag{84}$$

- How to calculate: Evaluating exactly $\sup_{k\geqslant n} x_k$ could be hard. There are two ways to overcome:

  ◦ Use Squeeze theorem:
  Instead of evaluating $\sup_{k\geqslant n} x_n$ exactly, find an upper bound and an lower bound

$$w_n \leqslant \sup \{x_n, \ldots\} \leqslant z_n \tag{85}$$

  and try to show $\lim w_n = \lim z_n$. Note that the requirement for z_n is $z_n \geqslant x_k$ for all $k \geqslant n$ while the requirement for w_n is $w_n \leqslant x_k$ for some $k \geqslant n$.

  Example 41. Consider $x_n = (-1)^n + e^{-n^2}$. We have

$$1 \leqslant \sup_{k\geqslant n} \left[(-1)^k + e^{-k^2} \right] \leqslant 1 + e^{-n^2}. \tag{86}$$

  Taking limit $n \longrightarrow \infty$ we conclude

$$\limsup x_n = 1. \tag{87}$$

  ◦ Use limsup is the largest limit of convergent subsequences. To take this approach we need to first guess $\limsup_{n\longrightarrow\infty} x_n = 1$. Then show

  1. There is a subsequence converging to 1. Take $n_k = 2\,k$ then $x_{n_k} = 1 + e^{-4k^2} \longrightarrow 1$.
  2. For every convergent subsequence $x_{n_k} \longrightarrow a$, $a \leqslant 1$. We do this through comparison theorem:

$$x_{n_k} = (-1)^{n_k} + e^{-n_k^2} \leqslant 1 + e^{-k^2} \Longrightarrow a = \lim_{k\longrightarrow\infty} x_{n_k} \leqslant \lim_{k\longrightarrow\infty} \left(1 + e^{-k^2}\right) = 1. \tag{88}$$

2. Infiinite Series

2.1. Definition and Basic Properties.

2.1.1. Definition.

Definition 42. (Infinite series) *Given a sequence* $\{a_n\}$ *of real numbers, the formal sum*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots \tag{89}$$

is called an "infinite series".

Remark 43. Up to now the "summation" of infinitely many real numbers

$$a_1 + a_2 + \cdots + a_n + \cdots \tag{90}$$

is "formal" because it is not clear what it means to say $a_1 + a_2 + \cdots + a_n + \cdots = s \in \mathbb{R}_{\text{ext}}$.

Remark 44. Note that $\sum_{n=1}^{\infty} a_n$ is **just another way** of denoting the formal sum $a_1 + a_2 + \cdots + a_n + \cdots$.

Example 45. Some examples of infinite series:

$$\sum_{n=1}^{\infty} (-1)^n = (-1) + 1 + (-1) + 1 + \cdots \tag{91}$$

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\sin 1}{1} + \frac{\sin 2}{2} + \cdots \tag{92}$$

$$\sum_{n=1}^{\infty} 2^n = 1 + 2 + 4 + \cdots \tag{93}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots \tag{94}$$

It is clear that the value of $\sum_{n=1}^{\infty} \frac{1}{2^n}$ should be 2 while $\sum_{n=1}^{\infty} 2^n$ should be ∞. It's not clear whether the first two sums corresponds to any value.

Definition 46. (Partial sum and convergence) *The nth partial sum of an infinite series* $\sum_{n=1}^{\infty} a_n$ *is defined as* $s_n = \sum_{m=1}^{n} a_m$*. If the sequence* $\{s_n\}$ *converges to some real number* s*, then we say the infinite series converges, and say its sum is* s*, and simply write*

$$\sum_{n=1}^{\infty} a_n = s. \tag{95}$$

If $s \longrightarrow \infty$ *or* $-\infty$*, we say the infinite series diverges to* ∞ *or* $-\infty$ *respectively and write*

$$\sum_{n=1}^{\infty} a_n = \infty \text{ or } -\infty. \tag{96}$$

Remark 47. There are many ways to assign a number s to an infinite sum $a_1 + a_2 + \cdots$. The above is the most popular one.

Recalling theorems for the convergence of sequences, we have

Theorem 48.

- $\sum_{n=1}^{\infty} a_n = s$ *if and only if for any* $\varepsilon > 0$*, there is* $N \in \mathbb{N}$ *such that for all* $n > N$*,*
$$\left| s - \sum_{m=1}^{n} a_m \right| < \varepsilon; \tag{97}$$
- $\sum_{n=1}^{\infty} a_n = \infty$ *if and only if for any* $M \in \mathbb{R}$*, there is* $N \in \mathbb{N}$ *such that for all* $n > N$*,*
$$\sum_{m=1}^{n} a_m > M; \tag{98}$$
- $\sum_{n=1}^{\infty} a_n = -\infty$ *if and only if for any* $M \in \mathbb{R}$*, there is* $N \in \mathbb{N}$ *such that for all* $n > N$*,*
$$\sum_{m=1}^{n} a_m < M. \tag{99}$$

Exercise 24. Prove the above theorem.

Theorem 49. (Cauchy) *A infinite series* $\sum_{n=1}^{\infty} a_n$ *converges to some* $s \in \mathbb{R}$ *if and only if for any* $\varepsilon > 0$ *there is* $N \in \mathbb{N}$ *such that for all* $m > n > N$*,*
$$\left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon. \tag{100}$$

Exercise 25. Prove the above theorem.

Corollary 50. *If* $\sum_{n=1}^{\infty} a_n$ *converges to* $s \in \mathbb{R}$ *then* $\lim_{n \longrightarrow \infty} a_n = 0$*. Equivalently, if* $\lim_{n \rightarrow \infty} a_n$ *does not exist, or exists but is not* 0*, then* $\sum_{n=1}^{\infty} a_n$ *does not converge to any real number.*

Proof. For any $\varepsilon > 0$, since $\sum_{n=1}^{\infty} a_n$ converges, it is Cauchy and there exists $N_1 \in \mathbb{N}$ such that for all $m > n > N_1$,
$$\left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon. \tag{101}$$

Now take $N = N_1 + 1$. Then for any $n > N$, we have $n > n - 1 > N_1$ which gives
$$|a_n - 0| = \left| \sum_{k=n}^{n} a_k \right| < \varepsilon. \tag{102}$$

Thus by definition of convergence of sequence $\lim_{n \longrightarrow \infty} a_n = 0$. □

Remark 51. The above corollary is very useful, however we should keep in mind that:

1. The converse is not true. That is $\lim_{n \rightarrow \infty} a_n = 0$ does not imply the convergence of $\sum_{n=1}^{\infty} a_n$.
2. It cannot be applied to conclude $\sum_{n=1}^{\infty} a_n \neq \infty$ or $-\infty$.

Exercise 26. Give a counterexample to the following claims:
$$\lim_{n \longrightarrow \infty} a_n = 0 \Longrightarrow \sum_{n=1}^{\infty} a_n = s \text{ for some } s \in \mathbb{R}. \tag{103}$$
$$\sum_{n=1}^{\infty} a_n = s \in \mathbb{R}_{\text{ext}} \Longrightarrow \lim_{n \longrightarrow \infty} a_n = 0. \tag{104}$$

Example 52. Let $a_n = r^{n-1}$ for $r \in \mathbb{R}$. Then

a) If $|r| < 1$, then $\sum_{n=1}^{\infty} a_n = 1 + r + r^2 + \cdots = \frac{1}{1-r}$.

b) If $r \geqslant 1$, then $\sum_{n=1}^{\infty} a_n = \infty$.

c) If $r \leqslant -1$, then $\sum_{n=1}^{\infty} a_n$ does not exist (as extended real number).

Proof.

a) We have

$$\sum_{m=1}^{n} a_m = 1 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}. \tag{105}$$

For any $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $N \geqslant \log_{|r|} [\varepsilon (1 - r)]$, then for any $n > N$,

$$\left| \frac{1}{1-r} - \sum_{m=1}^{n} a_m \right| = \frac{|r|^n}{1-r} < \frac{|r|^N}{1-r} < \varepsilon. \tag{106}$$

b) For any $M \in \mathbb{R}$. Take $N \in \mathbb{N}$ such that $N > |M|$. Then for every $n > N$ we have

$$\sum_{m=1}^{n} a_m \geqslant \sum_{m=1}^{n} 1 = n > N > |M| \geqslant M. \tag{107}$$

c) Since $r \leqslant -1$, $|a_n| \geqslant 1$. Therefore by Corollary 42 $\sum_{n=1}^{\infty} a_n$ does not converge to any real number. We still need to show that $\sum_{n=1}^{\infty} a_n \neq \infty, -\infty$. To do this, we show that $s_n = \sum_{m=1}^{n} a_n$ satisfies $s_n \geqslant 0$ when n is odd and $s_n \leqslant 0$ when n is even. Clearly $s_1 = 1 > 0$, $s_2 = 1 + r \leqslant 0$. For $n \geqslant 3$, calculate

$$s_n = \sum_{m=1}^{n-1} r^{m-1} + r^n = \frac{1 - r^{n-1}}{1 - r} + r^n. \tag{108}$$

As $r \leqslant -1$, $1 - r \geqslant 2$ which gives

$$\left| \frac{1 - r^{n-1}}{1 - r} \right| \leqslant \frac{1 + |r|^{n-1}}{2} \leqslant |r|^{n-1} \leqslant |r|^n. \tag{109}$$

Therefore, s_n and r^n cannot take opposite signs. □

Example 53. We have

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{1 - \frac{1}{2}} = 2; \qquad \sum_{n=1}^{\infty} \frac{1}{5^{n-1}} = \frac{5}{4}. \tag{110}$$

Example 54. Consider

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\sin 1}{1} + \frac{\sin 2}{2} + \cdots \tag{111}$$

We show that it actually converges. Consider the partial sum

$$S_m = \sum_{k=n+1}^{m} \frac{\sin k}{k}. \tag{112}$$

Denote $A_k = \sum_{l=1}^{k} \sin l$, and $B_k = \frac{1}{k}$. Then we have

$$\begin{aligned}\sum_{k=n+1}^{m} \frac{\sin k}{k} &= \sum_{k=n+1}^{m} (A_k - A_{k-1})\, B_k \\ &= [A_{n+1}\, B_{n+1} - A_n\, B_{n+1}] + [A_{n+2}\, B_{n+2} - A_{n+1}\, B_{n+2}] + \cdots \\ &\quad + [A_m\, B_m - A_{m-1}\, B_m] \\ &= [A_m\, B_m - A_n\, B_{n+1}] + \\ &\quad + [A_{n+1}\,(B_{n+1} - B_{n+2}) + A_{n+2}\,(B_{n+2} - B_{n+3}) + \cdots + A_{m-1}\,(B_{m-1} - B_m)] \\ &= [A_m\, B_m - A_n\, B_{n+1}] + \sum_{k=n+1}^{m-1} [A_k\,(B_k - B_{k+1})] \\ &= \left[\frac{A_m}{m} - \frac{A_n}{n+1}\right] + \sum_{k=n+1}^{m-1} \left[A_k \left(\frac{1}{k} - \frac{1}{k+1}\right)\right]. \end{aligned} \tag{113}$$

Now notice that $\{A_n\}$ is in fact a bounded sequence:

$$\begin{aligned} A_n &= \sin 1 + \sin 2 + \cdots + \sin n \\ &= \frac{\sin 1\, [\sin 1 + \sin 2 + \cdots + \sin n]}{\sin 1} \\ &= \frac{\cos (1-1) - \cos (1+1) + \cos (2-1) - \cos (2+1) + \cdots + \cos (n-1) - \cos (n+1)}{2 \sin 1} \\ &= \frac{[\cos 0 + \cos 1 + \cdots + \cos (n-1)] - [\cos 2 + \cos 3 + \cdots + \cos (n+1)]}{2 \sin 1} \\ &= \frac{\cos 0 + \cos 1 - \cos n - \cos (n+1)}{2 \sin 1}. \end{aligned} \tag{114}$$

Now it is clear that $|A_n| \leqslant \frac{2}{\sin 1}$ for all $n \in \mathbb{N}$.

Back to (113):

$$\begin{aligned} \left| \sum_{k=n+1}^{m} \frac{\sin k}{k} \right| &\leqslant \left| \frac{A_m}{m} \right| + \left| \frac{A_n}{n+1} \right| + \sum_{k=n+1}^{m-1} |A_k| \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &\leqslant \frac{2}{\sin 1} \left[\frac{1}{m} + \frac{1}{n+1} + \sum_{k=n+1}^{m-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) \right] \\ &= \frac{2}{\sin 1} \left[\frac{2}{n+1} \right] = \frac{4}{(n+1) \sin 1}. \end{aligned} \tag{115}$$

Now we are ready to show the series is Cauchy:

For any $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $N + 1 \geqslant \frac{4}{\varepsilon \sin 1}$, then for any $m > n > N$, we have

$$\left| \sum_{k=n+1}^{m} \frac{\sin k}{k} \right| \leqslant \frac{4}{(n+1) \sin 1} < \frac{4}{(N+1) \sin 1} \leqslant \varepsilon. \tag{116}$$

Therefore the series converges.

Exercise 27. Prove the following Abel summation formula: Let $A_k = \sum_{l=1}^{k} a_l$, $B_k = \sum_{l=1}^{k} b_l$, then

$$\sum_{k=n+1}^{m} a_k\, B_k = [A_m\, B_m - A_n\, B_{n+1}] - \sum_{k=n+1}^{m-1} A_k\, b_{k+1} \tag{117}$$

Draw analogy to the formula of integration by parts:

$$\int_a^b f(x)\, G(x)\, \mathrm{d}x = [F(b)\, G(b) - F(a)\, G(a)] - \int_a^b F(x)\, g(x)\, \mathrm{d}x \tag{118}$$

where $F(x) = \int_a^x f(t)\, \mathrm{d}t$, $G(x) = \int_a^x g(t)\, \mathrm{d}t$.

2.1.2. Arithmetics of infinite series.

Theorem 55. (Arithmetics) *If* $\sum_{n=1}^{\infty} a_n = s, \sum_{n=1}^{\infty} b_n = t$ *with* $s, t \in \mathbb{R}_{\text{ext}}$. *Then*

- *For any* $c \in \mathbb{R}$, *if* $c\,s$ *is defined,* $\sum_{n=1}^{\infty} (c\,a_n) = c\,s$.
- *If* $s + t$ *is defined,* $\sum_{n=1}^{\infty} (a_n + b_n) = s + t$.

Exercise 28. What happens if $c = \pm\infty$?

Exercise 29. Let $c = 0, s = \infty$ or $-\infty$. We know that $c\,s$ is not defined. However the sum $\sum_{n=1}^{\infty} (c\,a_n)$ is still well-defined. Explain why and find this sum.

Example 56. Let $a_n = c\,r^{n-1}$ for $r \in \mathbb{R}$ and $c \in \mathbb{R}$. Then

a) If $|r| < 1$, then $\sum_{n=1}^{\infty} a_n = 1 + r + r^2 + \cdots = \frac{c}{1 - r}$.

b) If $r \geqslant 1$ and $c \neq 0$, then $\sum_{n=1}^{\infty} a_n = c \cdot \infty = \begin{cases} \infty & c > 0 \\ -\infty & c < 0 \end{cases}$.

c) If $r \leqslant -1$ and $c \neq 0$, then $\sum_{n=1}^{\infty} a_n$ does not exist (as extended real number).

d) If $c = 0$ then $\sum_{n=1}^{\infty} a_n = 0$ no matter what value r takes.

a),b),d) clearly follow from Theorem 47. We prove c) by contradiction: If $\sum_{n=1}^{\infty} a_n = s \in \mathbb{R} \cup \{-\infty, \infty\}$, then since $c^{-1} \neq 0$, we have

$$\sum_{n=1}^{\infty} r^{n-1} = \sum_{n=1}^{\infty} (c^{-1}\,a_n) = c^{-1}\,s \in \mathbb{R} \cup \{-\infty, \infty\}. \tag{119}$$

Contradiction.

Example 57. We have

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{5^{n-1}} = \frac{1}{5}\frac{5}{4} = \frac{1}{4}. \tag{120}$$

Remark 58. In general there is no relation between $\sum_{n=1}^{\infty} a_n\,b_n$ and $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$. On the other hand, with some extra assumption we can define the product

$$\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = \sum_{n=1}^{\infty}\left[\sum_{k=1}^{n} a_k\ b_{n+1-k}\right] = (a_1\ b_1) + (a_1\ b_2 + a_2\ b_1) + (a_1\ b_3 + a_2\ b_2 + a_3\,b_1) + \cdots \tag{121}$$

This will be discussed in Math414.

2.1.3. Forbidden operations.

Example 59. (Grouping) Unless $a_n \geqslant 0$ (or all $\leqslant 0$) and $\sum_{n=1}^{\infty} a_n$ converges, the order of summation cannot be changed. For example let $a_n = (-1)^{n+1}$. If we are allowed to group terms together and sum them first, we would have both

$$\sum_{n=1}^{\infty} a_n = 1 + (-1) + 1 + \cdots = 1 + [(-1) + 1] + [(-1) + 1] + \cdots = 1 + 0 + 0 + \cdots = 1; \tag{122}$$

$$\sum_{n=1}^{\infty} a_n = 1 + (-1) + 1 + \cdots = [1 + (-1)] + [1 + (-1)] + \cdots = 0 + 0 + 0 + \cdots = 0. \tag{123}$$

Definition 60. (Rearrangement) *A rearrangement of an infinite series* $\sum_{n=1}^{\infty} a_n$ *is another infinite series* $\sum_{m=1}^{\infty} a_{n(m)}$ *where* $m{:}\mapsto n(m)$ *is a bijection from* $\mathbb{N}$ *to* $\mathbb{N}$.

Example 61. An example of rearrangement of $a_1 + a_2 + a_3 + \cdots$ is $a_2 + a_4 + a_7 + a_1 + a_5 + a_3 + a_6 + \cdots$.

Example 62. (Rearrangement) Consider the sequence $\sum_{n=1}^{\infty} a_n$ with $a_n = \frac{(-1)^{n+1}}{n}$. If we are allowed to freely rearrange (that is choose the order of summation), then for any $s \in \mathbb{R} \cup \{-\infty, \infty\}$, there is a rearrangement such that it converges to s.

Proof. Consider the case $s \in \mathbb{R}$. The cases $s = \infty, -\infty$ are left as exercises.

Consider the rearrangement $\sum_{n=1}^{\infty} b_n$ defined as follows:

- Let k_0 be such that $1 + \frac{1}{3} + \cdots + \frac{1}{2\,k_0 - 1} \geqslant s$ but $1 + \frac{1}{3} + \cdots + \frac{1}{2\,k_0 - 3} < s$. Set
$$b_1 = 1, b_2 = \frac{1}{3}, \ldots, b_{k_0} = \frac{1}{2\,k_0 - 1}; \tag{124}$$
The case $k_0 = 1$ is when $1 \geqslant s$. Then we just set $b_1 = 1$ and turn to the next step.
- Let k_1 be such that
$$\sum_{k=1}^{k_0} b_k - \left(\frac{1}{2} + \cdots + \frac{1}{2\,k_1 - 2} \right) \geqslant s, \qquad \sum_{k=1}^{k_0} b_k - \left(\frac{1}{2} + \cdots + \frac{1}{2\,k_1} \right) < s \tag{125}$$
and set
$$b_{k_0+1} = -\frac{1}{2}, \qquad b_{k_0+k_1} = -\frac{1}{2\,(k_1+1)}. \tag{126}$$
- Let k_2 be such that
$$\sum_{k=1}^{k_0+k_1} b_k + \left(\frac{1}{2\,k_0+1} + \cdots + \frac{1}{2\,k_0 + 2\,k_2 - 1} \right) \geqslant s, \sum_{k=1}^{k_0+k_1} b_k + \left(\frac{1}{2\,k_0+1} + \cdots + \frac{1}{2\,k_0 + 2\,k_2 - 3} \right) < s \tag{127}$$
and set
$$b_{k_0+k_1+1} = \frac{1}{2\,k_0+1}, \ldots, b_{k_0+k_1+k_2} = \frac{1}{2\,k_0 + 2\,k_2 + 1}, \tag{128}$$
- And so on.

Now set
$$S_l = \sum_{k=1}^{k_0+k_1+\cdots+k_l} b_k. \tag{129}$$
Then we see that if $n \in [k_0 + \cdots + k_l, k_0 + \cdots + k_{l+1}]$, then
$$s_n = \sum_{m=1}^{n} b_m \tag{130}$$
is always between S_l and S_{l+1}.

Finally notice that by construction, $|S_l - s| < \frac{1}{l}$. Thus for any $\varepsilon > 0$, take $L \in \mathbb{N}$ such that $L > \varepsilon^{-1}$. Now set $N = k_0 + \cdots + k_L$. For any $n > N$, there is $l \geqslant L$ such that $n \in [k_0 + \cdots + k_l, k_0 + \cdots + k_{l+1}]$. Therefore we have
$$|s_n - s| \leqslant \max\{|S_l - s|, |S_{l+1} - s|\} \leqslant \frac{1}{l} \leqslant \frac{1}{L} < \varepsilon. \tag{131}$$
That is $\sum_{n=1}^{\infty} b_n \longrightarrow s$ by definition. □

Remark 63. Note that the above proof depends on the fact that
$$\sum_{k=1}^{\infty} \frac{1}{2\,k-1} = \infty, \qquad \sum_{k=1}^{\infty} \left(-\frac{1}{2\,k} \right) = -\infty. \tag{132}$$

Exercise 30. Prove the above two facts.

2.2. Convergence tests.

2.2.1. Convergence/divergence through comparison.

In most situations, it is very hard or impossible to explicitly calculate the partial sum $S_n := \sum_{m=1}^{n} a_m$ and is therefore not possible to establish convergence/find the sum based on definition. Similar to the idea of Cauchy sequences, we need a way to determine the convergence of a series without obtaining explicitly the partial sums. This is possible thanks to the following theorem.

Theorem 64. *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two infinite series. Assume that there are $c > 0$ and $N_0 \in \mathbb{N}$ such that $|a_n| \leqslant c\, b_n$ for all $n > N_0$. Then*

a) $\sum_{n=1}^{\infty} b_n$ converges $\Longrightarrow$ $\sum_{n=1}^{\infty} a_n$ converges.

b) $\sum_{n=1}^{\infty} a_n$ does not converge to some real number $\Longrightarrow$ $\sum_{n=1}^{\infty} b_n = \infty$.

Proof. Note that a) and b) are equivalent logical statements, so we only need to prove a). We show that $\sum_{n=1}^{\infty} a_n$ is Cauchy. For any $\varepsilon > 0$, since $\sum_{n=1}^{\infty} b_n$ converges, it is Cauchy and there is $N_1 \in \mathbb{N}$ such that for all $m > n > N_1$,

$$\left| \sum_{k=n+1}^{m} b_k \right| < \frac{\varepsilon}{c}. \tag{133}$$

Take $N = \max\{N_1, N_0\}$. Then for any $m > n > N$,

$$\left| \sum_{k=n+1}^{m} a_k \right| \leqslant \sum_{k=n+1}^{m} |a_k| \leqslant c \left| \sum_{k=n+1}^{m} b_k \right| < \varepsilon. \tag{134}$$

So $\sum_{n=1}^{\infty} a_n$ is Cauchy and therefore converges. □

Example 65. It is clear by Theorem 56 that if $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} a_n$. The converse is not true, as can be seen from the following example:

Take $a_n = \frac{(-1)^{n+1}}{n}$. Then we clearly see that

$$S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2\,n-1} - \frac{1}{2\,n} = \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(2\,n-1)\,(2\,n)} \tag{135}$$

converges. On the other hand, we have $S_{2n+1} - S_{2n} \longrightarrow 0$ so S_{2n+1} converges to the same limit. From here it is easy to prove by definition that $S_n \longrightarrow$ to the same limit, which turns out to be $\ln 2$.

Remark 66. A sequence $\sum_{n=1}^{\infty} a_n$ that converges but with $\sum_{n=1}^{\infty} |a_n| = \infty$ is called conditionally convergent. It turns out that the phenomenon we have seen in Example 54 is quite generic for conditionally convergent series. More specifically, if $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then it can be re-arranged to converge to any extended real number.

On the other hand, a sequence $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} |a_n|$ converges is said to be *absolutely convergent*. Absolutely convergent sequences can undergo any re-arrangement and still converge to the same sum.

Exercise 31. Prove that if $\sum_{n=1}^{\infty} a_n$ does not converge to some $s \in \mathbb{R}$, then $\sum_{n=1}^{\infty} |a_n| = \infty$.

In light of the above theorem, it is important to study *non-negative* series, that is infinite series $\sum_{n=1}^{\infty} a_n$ satisfying $a_n \geqslant 0$ for all $n \in \mathbb{N}$. Once a non-negative sequence $\sum_{n=1}^{\infty} b_n$ is shown to be convergent, we know that any $\sum_{n=1}^{\infty} a_n$ satisfying $|a_n| \leqslant c\, b_n$ for some constant c is also convergent. It is further possible to make this comparison "intrinsic", that is design some criterion involving a_n only and guarantees the relation $|a_n| \leqslant c\, b_n$. Such criteria are usually called "tests". We will study the simpliest tests in the following section.

Exercise 32. Let $\sum_{n=1}^{\infty} a_n$ be a non-negative series. Then it converges $\Longleftrightarrow$ it is bounded above.

2.2.2. Typical non-negative series and their implications.

Ratio test and root test.

Both tests are based on the convergence/divergence of the Geometric series.

Example 67. (Geometric series) We have seen that $\sum_{n=1}^{\infty} r^{n-1}$ converges when $0 \leqslant r < 1$. As a consequence, if another series $\sum_{n=1}^{\infty} a_n$ satisfies

$$|a_n| \leqslant c\, r^{n-1} \tag{136}$$

for some $c > 0$ and for all $n >$ some $N_0 \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ converges.

The following two intrinsic "convergence tests" based on comparison with geometric series are the simplest and most popular tests for convergence/divergence.

Theorem 68. (Ratio test) *Let $\sum_{n=1}^{\infty} a_n$ a infinite series. Further assume that $a_n \neq 0$ for all $n \in \mathbb{N}$. Then*

- *If* $\limsup_{n\longrightarrow\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, *the series converges.*
- *If* $\liminf_{n\longrightarrow\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, *the series diverges.*

Proof.

- Assume $\limsup_{n\longrightarrow\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$. Set $r = \frac{L+1}{2}$ and $\varepsilon_0 = \frac{1-L}{2}$. By definition

$$\limsup_{n\longrightarrow\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\longrightarrow\infty} \left\{ \sup_{k\geqslant n} \left|\frac{a_{k+1}}{a_k}\right| \right\} \tag{137}$$

therefore there is $N \in \mathbb{N}$ such that for all $n > N$,

$$\left| \sup_{k\geqslant n} \left|\frac{a_{k+1}}{a_k}\right| - L \right| < \varepsilon_0 \tag{138}$$

which gives

$$\left| \sup_{n>N} \left|\frac{a_{n+1}}{a_n}\right| \right| < L + \varepsilon_0 = r < 1 \Longrightarrow 0 < \left|\frac{a_{n+1}}{a_n}\right| < r < 1. \tag{139}$$

This gives, for all $n > N+1$,

$$|a_n| < |a_{N+1}|\, r^{n-N-1} = \frac{|a_{N+1}|}{r^N}\, r^{n-1}. \tag{140}$$

Note that since N is fixed, we have

$$|a_n| < c\, r^{n-1} \tag{141}$$

for all $n > N$ and consequently $\sum_{n=1}^{\infty} a_n$ converges.

- Assume $\liminf_{n\longrightarrow\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$. Set $\varepsilon_0 = L - 1$. Then by definition, similar to the limsup case above, there is $N \in \mathbb{N}$ such that for all $n > N$,

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \tag{142}$$

which means for all $n \geqslant N + 1$

$$|a_n| \geqslant |a_{N+1}| \tag{143}$$

As a consequence $a_n \not\longrightarrow 0$. By Corollary 42 we know that $\sum_{n=1}^{\infty} a_n$ diverges. □

Example 69. Prove that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$ converges.

Proof. We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{\sqrt{n+1}} \tag{144}$$

therefore

$$\limsup_{n\longrightarrow\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1. \tag{145}$$

So the series converges. □

Exercise 33. Let $\sum_{n=1}^{\infty} a_n$ be a infinite series. Let $\sum_{n=1}^{\infty} b_n$ be the series obtained by dropping all 0's from $\{a_n\}$. Prove that

$$\sum_{n=1}^{\infty} a_n = s \Longleftrightarrow \sum_{n=1}^{\infty} b_n = s \tag{146}$$

for $s \in \mathbb{R}_{\text{ext}}$.

Exercise 34. Can the ratio test be modified to apply to $\sum_{n=1}^{\infty} a_n$ without assuming $a_n \neq 0$?

Theorem 70. (Root test) *Let $\sum_{n=1}^{\infty} a_n$ be a infinite series. Then*

- *If $\limsup_{n\longrightarrow\infty} |a_n|^{1/n} < 1$, then the series converges.*
- *If $\liminf_{n\longrightarrow\infty} |a_n|^{1/n} > 1$, then the series diverges.*

Proof.

- Assume $\limsup_{n\longrightarrow\infty} |a_n|^{1/n} = L < 1$. Set $r = \frac{L+1}{2}$ and $\varepsilon_0 = \frac{1-L}{2}$. Then by definition, as in the proof of the above ratio test, there is $N \in \mathbb{N}$ such that for all $n > N$,

$$|a_n|^{1/n} < r < 1 \Longrightarrow |a_n| < r^n. \tag{147}$$

Therefore $\sum_{n=1}^{\infty} a_n$ converges.

- Assume $\liminf_{n\longrightarrow\infty} |a_n|^{1/n} > 1$. The proof is left as exercise. □

Remark 71. In fact, for $x_n > 0$

$$\liminf_{n\longrightarrow\infty} \frac{x_{n+1}}{x_n} \leqslant \liminf_{n\longrightarrow\infty} (x_n)^{1/n} \leqslant \limsup_{n\longrightarrow\infty} (x_n)^{1/n} \leqslant \limsup_{n\longrightarrow\infty} \frac{x_{n+1}}{x_n}. \tag{148}$$

Therefore the root test is sharper than the ratio test, in the sense that any series that passes the ratio test for convergence will also pass the root test.

Example 72. Consider the infinite series $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\sum_{n=1}^{\infty} a_n$ where $a_n = \begin{cases} 2^{-k} & n=2k-1 \\ 3^{-k} & n=2k \end{cases}$. It can be easily verified that

$$\liminf_{n\longrightarrow\infty} \frac{a_{n+1}}{a_n}=0; \qquad \limsup_{n\longrightarrow\infty} \frac{a_{n+1}}{a_n}=\infty \tag{149}$$

so the ratio test does not apply. On the other hand

$$\limsup_{n\longrightarrow\infty} (a_n)^{1/n}=\frac{1}{\sqrt{2}} \tag{150}$$

so the root test tells us that the series converges.

Other tests.

Example 73. (Generalized harmonic series) The series $\sum_{n=1}^{\infty} \frac{1}{n^a}$ converges when $a>1$ and diverges when $a\leqslant 1$.

Proof. When $a\leqslant 1$, we have $\frac{1}{n^a}\geqslant\frac{1}{n}$ therefore it suffices to show the divergence of $\sum_{n=1}^{\infty}\frac{1}{n}$. We use the following trick:[6]

$$1 > \frac{1}{2}, \tag{151}$$

$$\frac{1}{2}+\frac{1}{3} > \frac{1}{4}+\frac{1}{4}=\frac{1}{2}, \tag{152}$$

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}, \tag{153}$$

$$\vdots \quad \vdots \quad \vdots$$

Therefore we have (recall s_n is the partial sum)

$$s_{n_k}>\frac{k+1}{2} \tag{154}$$

where $n_k=1+2+2^2+\cdots+2^k=2^{k+1}-1$. Therefore s_n is not bounded and the series diverges to ∞.

When $a>1$, we use the following trick:

$$1 \leqslant 1; \tag{155}$$

$$2^{-a}+3^{-a} < 2\cdot 2^{-a}=2^{1-a}; \tag{156}$$

$$4^{-a}+5^{-a}+6^{-a}+7^{-a} < 4\cdot 4^{-a}=2^{2(1-a)}; \tag{157}$$

$$\vdots \quad \vdots \quad \vdots$$

We see that

$$\sum_{n=1}^{N}\frac{1}{n^a}<\sum_{n=1}^{\infty}\left(2^{(1-a)}\right)^{n-1}=\frac{1}{1-2^{1-a}}<+\infty. \tag{158}$$

6. This proof is attributed to Nicole Oresme (c1320-1325 – 1382).

Thus this non-negative series is bounded from above and therefore converges. □

Exercise 35. Let $\{x_n\}$ be a sequence. Prove: $\{x_n\}$ is bounded $\Longleftrightarrow$ Every subsequence of $\{x_n\}$ is bounded.

Remark 74. The standard proof for $a > 1$, usually presented after introduction of integration, is as follows:

$$\frac{1}{n^a} = \int_{n-1}^{n} \frac{1}{n^a} \mathrm{d}x \leqslant \int_{n-1}^{n} \frac{\mathrm{d}x}{x^a}. \tag{159}$$

Thus

$$\sum_{n=1}^{N} \frac{1}{n^a} = 1 + \sum_{n=2}^{N} \frac{1}{n^a} \leqslant 1 + \sum_{n=2}^{N} \int_{n-1}^{n} \frac{\mathrm{d}x}{x^a} = 1 + \int_{1}^{N} \frac{\mathrm{d}x}{x^a} = 1 + \frac{1 - N^{1-a}}{a-1} \leqslant \frac{a}{a-1}. \tag{160}$$

Remark 75. Recall that weeks ago we proved the convergence of the $a = 2$ case through the trick $\frac{1}{n\,(n-1)} \geqslant \frac{1}{n^2}$, however for more exotic a such tricks are not available anymore.

Remark 76. Also note that neither the ratio test nor the root test works for the generalized harmonic series.

Remark 77. It is also possible to design convergence/divergence tests using generalized harmonic series as the gauge. Since $\frac{1}{n^a}$ converges to 0 slower than r^n (in the sense that $\lim_{n\longrightarrow\infty} n^a\, r^n = 0$ if $|r| < 1$), these tests will be more refined than either the ratio test or the root test. One of such test is the following

(Raabe's test) $a_n > 0$. Then

- $\sum_{n=1}^{\infty} a_n$ converges if
$$\liminf_{n\longrightarrow\infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right) > 1 \tag{161}$$
- $\sum_{n=1}^{\infty} a_n$ diverges if
$$\limsup_{n\longrightarrow\infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right) < 1. \tag{162}$$

Note that the "convergent" part of Raabe's test can be turned into a convergence test for general series (without requiring $a_n > 0$) as

$$\liminf_{n\longrightarrow\infty} n \left(\frac{|a_n|}{|a_{n+1}|} - 1 \right) > 1 \Longrightarrow \sum_{n=1}^{\infty} a_n \text{ converges} \tag{163}$$

but the divergent part cannot, that is

$$\limsup_{n\longrightarrow\infty} n \left(\frac{|a_n|}{|a_{n+1}|} - 1 \right) < 1 \Longrightarrow \sum_{n=1}^{\infty} a_n \text{ diverges} \tag{164}$$

is not true due to the existence of convergent sequences such as $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

If we consider even slower convergent series, such as

$$\sum_{n=1}^{\infty} \frac{1}{n\,(\log n)^{\alpha}}, \qquad \sum_{n=1}^{\infty} \frac{1}{n\,(\log n)\,(\log\,(\log n))\cdots(\log\,(\cdots\log n))^{\alpha}} \tag{165}$$

for $\alpha > 1$ (the proof of convergence of these series is similar to that of the generalized harmonic series), we will obtain even sharper tests (Gauss' test from the former, Bertrand's test from the latter), but the formulas become quite baroque.

Finally let's look at a few fun examples.

Example 78. We know that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞. Now consider the sequence obtained by dropping all terms involving 9:

$$1 + \frac{1}{2} + \cdots + \frac{1}{8} + \cancel{\frac{1}{9}} + \frac{1}{10} + \cdots + \frac{1}{18} + \cancel{\frac{1}{19}} + \cdots + \frac{1}{88} + \cancel{\frac{1}{89}} + \cancel{\frac{1}{90}} + \cancel{\frac{1}{91}} + \cancel{\frac{1}{92}} + \cdots \tag{166}$$

Does this series converge?

As it is a non-negative sequence, we only need to check whether it's bounded above (Theorem 59). We have,

$$1 + \cdots + \frac{1}{8} < 8, \tag{167}$$

$$\left(\frac{1}{10} + \cdots + \frac{1}{18}\right) + \cdots + \left(\frac{1}{80} + \cdots \frac{1}{88}\right) < 8 \cdot \frac{9}{10} \tag{168}$$

In general, there are $8 \cdot 9^k$ terms between 10^{-k} and $\frac{1}{10^{k+1} - 1}$, so their sum is bounded above by $8 \cdot \left(\frac{9}{10}\right)^k$. Overall the sum is bounded above by

$$\sum_{n=0}^{\infty} 8 \cdot \left(\frac{9}{10}\right)^n = 8 \cdot \frac{1}{1 - \frac{9}{10}} = 80. \tag{169}$$

Therefore the new series converges.

Remark 79. Obviously we can try to study the sequence resulted from deleting all terms involving other digits, or sequence of numbers. For example we can delete all terms involving the combination 43, that is $\frac{1}{4352}$ is deleted while $\frac{1}{4537}$ is not. We can even play some silly games such as deleting all terms involving 121221, or someone's birthday. The resulting sequences are all convergent.

Example 80. (Fermat's Last Theorem) This is adapted from the blog of Terence Tao of UCLA[7]. We all know that Fermat's Last Theorem claims that

$$x^n + y^n = z^n, \qquad x, y, z \in \mathbb{N} \tag{170}$$

does not have any solution when $n \geqslant 3$. On the other hand, it is well-known that when $n = 2$, there are infinitely many solutions. But why? What's the difference between $n = 2$ and $n > 2$? It turns out that we can reveal some difference through knowledge of convergence/divergence of infinite series.

Let's consider the chance of three numbers $a, b, a + b$ are all the nth power of a natural number. If we treat a as a typical number of size a, then it's chance of being an nth power is roughly $a^{1/n}/a$. Ignoring the relation between $a, b, a + b$, we have the following probability for $a, b, a + b$ solving the equation (170):

$$a^{\frac{1}{n} - 1} b^{\frac{1}{n} - 1} (a + b)^{\frac{1}{n} - 1}. \tag{171}$$

7. The probabilistic heuristic justification of the ABC conjecture, link at http://terrytao.wordpress.com/2012/09/18/the-probabilistic-heuristic-justification-of-the-abc-conjecture/

Now consider all numbers a, b, we sum up the probabilities:

$$I := \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \left[a^{\frac{1}{n}-1} b^{\frac{1}{n}-1} (a+b)^{\frac{1}{n}-1} \right] \tag{172}$$

and apply the following intuition based on the so-called Borel-Cantelli Lemma in probability:

> If $I < \infty$, then the chance of (170) having a solution is very low, while if $I = \infty$, the chance is very high.

We notice that I has perfect symmetry between a and b, which means

$$I = 2 \sum_{a=1}^{\infty} \sum_{b=1}^{a-1} \left[a^{\frac{1}{n}-1} b^{\frac{1}{n}-1} (a+b)^{\frac{1}{n}-1} \right] + \sum_{a=1}^{\infty} a^{\frac{2}{n}-2} (2\,a)^{\frac{1}{n}-1}. \tag{173}$$

It is clear that the second series converges for all $n \geqslant 2$ so can be ignored for our purpose.

Now consider

$$J = \sum_{a=1}^{\infty} \sum_{b=1}^{a-1} \left[a^{\frac{1}{n}-1} b^{\frac{1}{n}-1} (a+b)^{\frac{1}{n}-1} \right]. \tag{174}$$

When $1 \leqslant b \leqslant a-1$, we have $a < a+b < 2\,a$ therefore

$$\sum_{a=1}^{\infty} \sum_{b=1}^{a-1} a^{\frac{1}{n}-1} b^{\frac{1}{n}-1} (2\,a)^{\frac{1}{n}-1} < J < \sum_{a=1}^{\infty} \sum_{b=1}^{a-1} a^{\frac{1}{n}-1} b^{\frac{1}{n}-1} (a)^{\frac{1}{n}-1} \tag{175}$$

which gives

$$2^{\frac{1}{n}-1} \sum_{a=1}^{\infty} \sum_{b=1}^{a-1} a^{\frac{2}{n}-2} b^{\frac{1}{n}-1} < J < \sum_{a=1}^{\infty} \sum_{b=1}^{a-1} a^{\frac{2}{n}-2} b^{\frac{1}{n}-1}. \tag{176}$$

So finally all we need to study is the convergence/divergence of

$$K = \sum_{a=1}^{\infty} \sum_{b=1}^{a-1} a^{\frac{2}{n}-2} b^{\frac{1}{n}-1} = \sum_{a=1}^{\infty} \left[a^{\frac{2}{n}-2} \sum_{b=1}^{a-1} b^{\frac{1}{n}-1} \right]. \tag{177}$$

We have

$$\sum_{b=1}^{a-1} b^{\frac{1}{n}-1} \sim \int_1^{a-1} x^{\frac{1}{n}-1} \mathrm{d}x \sim a^{\frac{1}{n}}. \tag{178}$$

Therefore

$$K \sim \sum_{a=1}^{\infty} a^{\frac{3}{n}-2} \tag{179}$$

which is convergent when $n \geqslant 4$ while divergent when $n = 2$.

Remark 81. The case $n = 3$ is a bit tricky here. The series $\sum_{a=1}^{\infty} a^{\frac{3}{n}-2}$ becomes the Harmonic series $\sum_{a=1}^{\infty} a^{-1}$ which is the borderline between convergence and divergence. Our argument does not provide any insight on why $x^3 + y^3 = z^3$ should not have solutions.

3. LIMIT OF FUNCTIONS

3.1. Limit of functions at $x_0 \in \mathbb{R}$.

Definition 82. *We say that a real number L is the limit of f at x_0, denoted $\lim_{x \to x_0} f(x) = L$ (or $f(x) \longrightarrow L$ as $x \longrightarrow x_0$, or $\lim f(x) = L$ as $x \longrightarrow x_0$), if*

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that for all } 0 < |x - x_0| < \delta \Longrightarrow |f(x) - L| < \varepsilon. \tag{180}$$

Remark 83. Note that the requirement $0 < |x - x_0|$ is important. The limit of f at x_0 has **nothing** to do with whether $f(x_0)$ is defined or not, not to say its value. This is very reasonable: Consider $f(x) = 0$ everywhere except $f(x) = 1$ at $x = 0$. Then clearly we should have $\lim_{x \longrightarrow 0} f(x) = 0$.

Exercise 36. If we use $0 \leqslant |x - x_0| < \delta$ instead, then what can we say about $\lim_{x \longrightarrow 0} f(x)$ for the above $f(x)$?

Example 84. The following hold:

a) $\lim_{x \longrightarrow x_0} a = a$ for constant function a.

b) $\lim_{x \longrightarrow 1} x^2 = 1$.

Proof. We prove b).

Given any $\varepsilon > 0$, we need to find $\delta > 0$ such that for all $0 < |x - 1| < \delta$, $|x^2 - 1| < \varepsilon$. Since

$$|x^2 - 1| = |x - 1|\,|x + 1| < \delta\,|x + 1| < \delta\,(2 + \delta). \tag{181}$$

we see that we need to choose a δ such that $\delta\,(2 + \delta) < \varepsilon$.

Notice that

$$\delta\,(2 + \delta) < \varepsilon \Longleftrightarrow \delta^2 + 2\,\delta + 1 < \varepsilon + 1 \Longleftrightarrow |\delta + 1| < \sqrt{\varepsilon + 1} \Longleftrightarrow -1 - \sqrt{\varepsilon + 1} < \delta < \sqrt{\varepsilon + 1} - 1. \tag{182}$$

Thus for any $\varepsilon > 0$, if we take $\delta = \frac{\sqrt{\varepsilon + 1} - 1}{2}$, then for $0 < |x - 1| < \delta$, $|x^2 - 1| < \varepsilon$. □

3.2. Limit of functions at $\pm\infty$ and $\pm\infty$ as limits.

Definition 85.

- *We say $\lim_{x \longrightarrow \infty} f(x) = L \in \mathbb{R}$ if for every $\varepsilon > 0$ there is some $M \in \mathbb{R}$ such that $x > M \Longrightarrow |f(x) - L| < \varepsilon$. The limit $\lim_{x \longrightarrow -\infty} f(x) = L$ is defined similarly.*
- *Let $x_0 \in \mathbb{R}$. We say $\lim_{x \longrightarrow x_0} f(x) = +\infty$ iff for every $M \in \mathbb{R}$ there is $\delta > 0$ such that*

$$|x - x_0| < \delta \Longrightarrow f(x) > M. \tag{183}$$

  $\lim_{x \longrightarrow x_0} f(x) = -\infty$ *can be defined similarly.*
- *We say $\lim_{x \longrightarrow +\infty} f(x) = +\infty$ iff for every $M \in \mathbb{R}$ there is $K \in \mathbb{R}$ such that*

$$x > K \Longrightarrow f(x) > M. \tag{184}$$

  The other three cases are similarly defined.

Example 86. Prove that $\ln(1/|x|) \longrightarrow \infty$ as $x \longrightarrow 0$.

Proof. For any $M \in \mathbb{R}$, we need to find $\delta > 0$ such that when $0 < |x| < \delta$, $\ln(1/|x|) > M$. It is clear that we can take $\delta = e^{-M-1}$. □

Example 87. Find and prove the limit (only discuss $x>0$)

$$\lim_{x\longrightarrow\infty}\left(\sqrt{x^2+2\,x}-x\right). \tag{185}$$

Solution.

The function is quite complicated and the limit is not obvious. Thus we first try to simplify:

$$\sqrt{x^2+2\,x}-x=\frac{\left(\sqrt{x^2+2\,x}-x\right)\left(\sqrt{x^2+2\,x}+x\right)}{\left(\sqrt{x^2+2\,x}+x\right)}=\frac{2\,x}{\sqrt{x^2+2\,x}+x}=\frac{2}{\sqrt{1+\frac{2}{x}}+1}. \tag{186}$$

All we need to do is to find the limits of the numerator and denominator. Clearly

$$\lim_{x\longrightarrow\infty}2=2. \tag{187}$$

On the other hand we show $\lim_{x\longrightarrow\infty}\sqrt{1+2/x}=1$. For any $\varepsilon>0$, take $M=1/\varepsilon$. Then for all $x>M$, we have $2/x<2\,\varepsilon$ and consequently

$$0<\sqrt{1+2/x}-1<\sqrt{1+2\,\varepsilon}-1<\sqrt{(1+\varepsilon)^2}-1=\varepsilon. \tag{188}$$

Therefore $\left|\sqrt{1+2/x}-1\right|<\varepsilon$ for all $x>M$. We have proved $\lim_{x\longrightarrow\infty}\left(\sqrt{1+2/x}+1\right)=2\neq 0$.

Now we can apply Theorem ? once more to conclude

$$\lim_{x\longrightarrow\infty}\left(\sqrt{x^2+2\,x}-x\right)=\lim_{x\longrightarrow\infty}\frac{2}{\sqrt{1+\frac{2}{x}}+1}=1. \tag{189}$$

3.3. Left and right limits.

Definition 88. (Left and right limits) *Let $f(x)$ be a real function and $x_0\in\mathbb{R}$. We say L is the left-limit of f at x_0 if*

$$\forall\varepsilon>0\ \exists\delta>0 \text{ such that for all } -\delta<x-x_0<0,\qquad |f(x)-L|<\varepsilon. \tag{190}$$

We denote it by

$$\lim_{x\longrightarrow x_0-}f(x)=L; \tag{191}$$

We say L is a right-limit of f at x_0 if

$$\forall\varepsilon>0\ \exists\delta>0 \text{ such that for all } 0<x-x_0<\delta,\qquad |f(x)-L|<\varepsilon. \tag{192}$$

We denote it by

$$\lim_{x\longrightarrow x_0+}f(x)=L. \tag{193}$$

Exercise 37. Prove that

$$\lim_{x\longrightarrow x_0}f(x)=L\in\mathbb{R}_{\text{ext}}\iff\lim_{x\longrightarrow x_0-}f(x)=L \text{ and } \lim_{x\longrightarrow x_0+}f(x)=L. \tag{194}$$

3.4. Properties of function limits.

First we establish relation between function limits and sequence limits.

Theorem 89. *Let $x_0,L\in\mathbb{R}_{\text{ext}}$. Then $\lim_{x\longrightarrow x_0}f(x)=L$ if and only if for every sequence $x_n\longrightarrow x_0$ with $x_n\neq x_0$ for all $n\in\mathbb{N}$, $\lim_{n\longrightarrow\infty}f(x_n)=L$.*

Proof. We prove “if” ($\Longleftarrow$) and then “only if” ($\Longrightarrow$),

- ”If”. We need to show if for every sequence $x_n \longrightarrow x_0$ with $x_n \neq x_0$ for all $n \in \mathbb{N}$, $\lim_{n\longrightarrow\infty} f(x_n) = L$ then $\lim_{x\longrightarrow x_0} f(x) = L$. We prove by contradiction.

  Assume the contrary. That is assume $f(x)$ does not have limit L as $x \longrightarrow x_0$. Checking Definition 6 we see that “$f(x)$ does not have limit L as $x \longrightarrow x_0$” means

  $$\exists\varepsilon_0\ \forall\delta\ \exists x \text{ with } 0 < |x - x_0| < \delta, \qquad |f(x) - L| > \varepsilon_0. \tag{195}$$

  Take $\delta = 1$. We have x_1 with $0 < |x_1 - x_0| < 1$, such that

  $$|f(x_1) - L| > \varepsilon_0; \tag{196}$$

  Next take $\delta = 1/2$. We have x_2 with $0 < |x_2 - x_0| < 1/2$, such that

  $$|f(x_2) - L| > \varepsilon_0; \tag{197}$$

  Continue doing this, we obtain a sequence $\{x_n\}$ satisfying $0 < |x_n - x_0| < 1/n$,

  $$|f(x_n) - L| > \varepsilon_0. \tag{198}$$

  Clearly $x_n \neq x_0$ for all $n \in \mathbb{N}$ by construction. To reach contradiction, we prove

  - $x_n \longrightarrow x_0$. For any $\varepsilon > 0$, choose natural number $N > 1/\varepsilon$. Then for every $n > N$, we have

    $$|x_n - x_0| < 1/N < \varepsilon. \tag{199}$$

  - $f(x_n) \not\longrightarrow L$. Take $\varepsilon = \varepsilon_0$. Since by construction $|f(x_n) - L| > \varepsilon_0$ for all $n \in \mathbb{N}$, there does not exists $N \in \mathbb{N}$ such that $n > N \Longrightarrow |f(x_n) - L| < \varepsilon$.

  Thus we have shown that there is a sequence $x_n \longrightarrow x_0$ with $x_n \neq x_0$ for all $n \in \mathbb{N}$ but $\lim_{n\longrightarrow\infty} f(x_n) = L$ does not hold. Contradiction.

- ”Only if”. We need to show that if $\lim_{x\longrightarrow x_0} f(x) = L$ then for every sequence $x_n \longrightarrow x_0$ with $x_n \neq x_0$ for all $n \in \mathbb{N}$, $\lim_{n\longrightarrow\infty} f(x_n) = L$. We prove directly this time.

  Let $\{x_n\}$ be a sequence satisfying $x_n \longrightarrow x_0$ with $x_n \neq x_0$ for all $n \in \mathbb{N}$. Given any $\varepsilon > 0$, since $\lim_{x\longrightarrow x_0} f(x) = L$ there is $\delta > 0$ such that $0 < |x - x_0| < \delta \Longrightarrow |f(x) - L| < \varepsilon$.

  Now for this δ, we have $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - x_0| < \delta$ because $x_n \longrightarrow x_0$. Thus for all $n > N$,$|f(x_n) - L| < \varepsilon$.

  Putting the above together, we see that for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$, $|f(x_n) - L| < \varepsilon$ which means $\lim_{n\longrightarrow\infty} f(x_n) = L$. □

Remark 90. This theorem is very useful in proving non-existence of limit.

Example 91. Prove that $\lim_{x\longrightarrow 0} \cos\left(\frac{1}{x}\right)$ does not exist.

Proof. Take $x_n = \frac{1}{2n\pi}$, $y_n = \frac{1}{(2n+1)\pi}$. Then

$$\forall n \in \mathbb{N}, \qquad x_n \neq 0, y_n \neq 0; \tag{200}$$

$$\lim_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} y_n = 0; \tag{201}$$

But

$$\lim_{n\longrightarrow\infty}\cos\left(\frac{1}{x_n}\right)=\lim_{n\longrightarrow\infty}1=1 \text{ while } \lim_{n\longrightarrow\infty}\cos\left(\frac{1}{y_n}\right)=\lim_{n\longrightarrow\infty}(-1)=-1. \tag{202}$$

Therefore the original limit does not exsit. □

Using the above theorem, the following are easy to prove and are left as exercises.

Theorem 92. *Let $f\colon E\subseteq\mathbb{R}_{\text{ext}}\mapsto\mathbb{R}_{\text{ext}}$. Let $x_0\in\mathbb{R}_{\text{ext}}$ be a limit point of E. Let $L, M\in\mathbb{R}_{\text{ext}}$. If $\lim_{x\to x_0}f(x)=L, \lim_{x\to x_0}g(x)=M$, then, as long as the right hand sides are well-defined,*

$$\lim_{x\to x_0}(f\pm g)(x)=L\pm M, \qquad \lim_{x\to x_0}(f g)(x)=L\,M, \tag{203}$$

and, if furthermore $M\neq 0$,

$$\lim_{x\to x_0}\left(\frac{f}{g}\right)(x)=\frac{L}{M}. \tag{204}$$

Theorem 93. (Comparison) *If $h(x)\leqslant f(x)\leqslant g(x)$, and $\lim_{x\to x_0}h(x)=L_1, \lim_{x\to x_0}f(x)=L_2, \lim_{x\to x_0}g(x)=L_3$, then $L_1\leqslant L_2\leqslant L_3$.*

Theorem 94. (Squeeze) *If $h(x)\leqslant f(x)\leqslant g(x)$, $\lim_{x\to x_0}h(x)=\lim_{x\to x_0}g(x)=L$, then $\lim_{x\to x_0}f(x)=L$.*

Lemma 95. *A function can have at most one limit at a given point.*

Proof. We prove by contradiction. Assume that $f(x)\longrightarrow L_1$ and $f(x)\longrightarrow L_2$ as $x\longrightarrow x_0$. Take $\varepsilon=|L_1-L_2|/2$. From the first limit we conclude that there is $\delta_1>0$ such that for all $0<|x-x_0|<\delta_1$, $|f(x)-L_1|<\varepsilon$; Similarly from the second limit we conclude that there is $\delta_2>0$ such that for all $0<|x-x_0|<\delta_2$, $|f(x)-L_2|<\varepsilon$.

Now take $\delta=\min\{\delta_1, \delta_2\}$. For every x such that $|x-x_0|<\delta$, we have at the same time $|f(x)-L_1|<\varepsilon$ and $|f(x)-L_2|<\varepsilon$. This leads to

$$|L_1-L_2|=|(f(x)-L_1)-(f(x)-L_2)|\leqslant|f(x)-L_1|+|f(x)-L_2|<2\,\varepsilon=|L_1-L_2|. \tag{205}$$

Contradiction. □

Example 96. Prove that $\lim_{x\longrightarrow 0}x\sin(1/x)=0$.

Proof. We take $h(x)=-|x|, g(x)=|x|, f(x)=x\sin(1/x)$. Clearly $h(x)\leqslant f(x)\leqslant g(x)$. Thus all we need to show is $h(x), g(x)\longrightarrow 0$. Further observe that $h(x)=-g(x)$, by Theorem ? and the fact that $\lim_{x\longrightarrow 0}(-1)=-1$, we have $\lim_{x\longrightarrow 0}h(x)=-\lim_{x\longrightarrow 0}g(x)$.

Therefore all we need to show is $\lim_{x\longrightarrow 0}|x|=0$. For every $\varepsilon>0$, take $\delta=\varepsilon$, we have for all $|x-0|<\delta$,

$$||x|-0|=|x|<\delta=\varepsilon. \tag{206}$$

Thus ends the proof. □

4. UNIFYING DIFFERENT LIMITS – A BRIEF INTRODUCTION TO TOPOLOGY

As we have seen, there are many seemingly different definitions of limits: limit of a sequence, limit of a function at a point, limit of a function at $+\infty$, limit is a real number, limit is $\pm\infty$, and many more. However, closer inspection reveals that they all have the same format: Limit of real function. $f(x) \longrightarrow L$ as $x \longrightarrow a$ is defined as

- $\boldsymbol{a \in \mathbb{R}, L \in \mathbb{R}}$. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall 0 < |x - a| < \delta$, $|f(x) - L| < \varepsilon$.
- $\boldsymbol{a \in \mathbb{R}, L = \infty}$. $\forall M \in \mathbb{R}, \exists \delta > 0$ such that $\forall 0 < |x - a| < \delta$, $f(x) > M$.
- $\boldsymbol{a \in \mathbb{R}, L = -\infty}$. $\forall M \in \mathbb{R}, \exists \delta > 0$ such that $\forall 0 < |x - a| < \delta$, $f(x) < M$.
- $\boldsymbol{a = \infty, L \in \mathbb{R}}$. $\forall \varepsilon > 0$, $\exists M \in \mathbb{R}$ such that $\forall x > M$, $|f(x) - L| < \varepsilon$.
- $\boldsymbol{a = -\infty, L = \infty}$. $\forall M \in \mathbb{R}$, $\exists M' \in \mathbb{R}$ such that $\forall x < M'$, $f(x) > M$. Note that M and M' are not the same number.

The "difference" only arises because of different quantification of $x \longrightarrow a$ (red) and $f(x) \longrightarrow L$ (orange). In other words, once we quantify what it means to say "approaching an extended real number", all definitions will naturally follow. In mathematical jargon, we need to give $\mathbb{R}_{\text{ext}}$ a *topology*.

Definition 97. (Topology) *Let X be a set. A "topology" on X is X together with a collection S of certain subsets of X satisfying*

1. $X, \phi \in S$;
2. *The union of any number of elements in S is still in S.*
3. *The intersection of finitely many elements in S is still in S.*

Exercise 38. Let $X = \mathbb{R}$ and S be the collection of all open sets. Prove that this is a topology.

Exercise 39. Let $X = \mathbb{R}$ and S be the collection of all closed sets. Prove that this is not a topology.

Once a topology is defined, we can give a unified definition of limit as follows.

Definition 98. (Unified definition of limit) *Let (X, S) be a topology and (Y, T) be another. Let $f(x): X \mapsto Y$. Let $x_0 \in X, L \in Y$. Then $\lim_{x \longrightarrow x_0} f(x) = L$ is defined through*

> *For any open set $A \in T$ with $L \in A$, there is an open set $B \in S$ such that for all $x \in B$, $x \neq x_0$, $f(x) \in A$.*

Exercise 40. Prove that the above definition is equivalent to

- For any open set $A \in T$ with $L \in A$, there is an open set $B \in S$ such that $f(B) \subseteq A$.

Exercise 41. Prove that $f(x): X \mapsto Y$ is continuous if and only if

> The preimage of any open set is open.

Show that this claim cannot be adapted to characterize the continuity of $f(x)$ at (one point) x_0.

Exercise 42. Let $X = \mathbb{R}_{\text{ext}}$. Let $S = \{(a, b) | a, b \in \mathbb{R}, a < b\} \cup \{(a, \infty] | a \in \mathbb{R}\} \cup \{[-\infty, b) | b \in \mathbb{R}\}$.

a) Prove that this is a topology.

b) Prove that with this topology all our definitions of limit of function can be unified under Definition 98.

Exercise 43. Let (X, S) be a topology. Let $M \subset X$. Define $T = \{A \cap M | A \in S\}$.

a) Prove that (M, T) is a topology.

b) Can you unify the definitions of sequence limit?

5. Continuity

5.1. Definitions.

Definition 99. (Continuity) *A function $f(x): \mathbb{R} \mapsto \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if and only if*

1. $\lim_{x \longrightarrow x_0} f(x)$ *exists;*
2. $f(x_0) = \lim_{x \longrightarrow x_0} f(x)$.

Or equivalently, if

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that for all } |x - x_0| < \delta, \qquad |f(x) - f(x_0)| < \varepsilon. \tag{207}$$

If $f(x)$ is continuous at every $x \in \mathbb{R}$, we say $f(x)$ is continuous on $\mathbb{R}$.

Remark 100. Note that compared with definition of limit of function, we do not require $0 < |x - x_0|$ anymore.

Exercise 44. Give definition for left and right continuity at $x_0 \in \mathbb{R}$.

Definition 101. (Continuity on an interval) *A function $f(x): [a, b] \mapsto \mathbb{R}$ is said to be continuous on the closed interval $[a, b]$ if and only if all the following are satisfied*

- *$f(x)$ is continuous at every $x \in (a, b)$;*
- *$f(x)$ is left continuous at b;*
- *$f(x)$ is right continuous at a.*

Exercise 45. Give definitions of continuity for $f(x): (a, b) \mapsto \mathbb{R}$, $f(x): [a, b) \mapsto \mathbb{R}, f(x): (a, b[\mapsto \mathbb{R}$.

Exercise 46. Let $E \subseteq \mathbb{R}$ be any subset of $\mathbb{R}$. Let $f(x): E \mapsto \mathbb{R}$. Try to give definition for "$f(x)$ is continuous on E". Do you encounter any new difficulty?

Remark 102. We simply say "f is continuous" when f is continuous on its domain.

Example 103. Let $f_1(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Prove that it is continuous at $x = 0$.

Proof. Take any $\varepsilon > 0$ we need to find $\delta > 0$ such that for every $|x - 0| < \delta$, $|f_1(x) - 0| < \varepsilon$. We take $\delta = \left(\frac{1}{\log(1/\varepsilon)} \right)^{1/2}$. To show that this δ works, we need to discuss two cases (be careful here!!):

- $x = 0$. In this case $|0 - 0| < \varepsilon$;
- $x \neq 0$. In this case $|f_1(x) - 0| = e^{-1/x^2} < e^{-1/\delta^2} = \varepsilon$. □

Example 104. Let $f_2(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & |x| \geqslant 1 \end{cases}$. Prove that it is continuous at $x = 1$.

Proof. Take any $\varepsilon > 0$ we need to find $\delta > 0$ such that for every $|x - 1| < \delta$, $|f_2(x) - 0| < \varepsilon$. Take $\delta = \ln(1/\varepsilon)$, then for every $|x - 1| < \delta$ there are two cases:

- $x \geqslant 1$. In this case $|f_2(x) - f_2(1)| = 0 < \varepsilon$.
- $x < 1$. In this case

$$|f_2(x) - f_2(1)| = e^{-\frac{1}{(1-x)(1+x)}} < e^{-\frac{1}{\delta}} = \varepsilon. \tag{208}$$

Thus ends the proof. □

Remark 105. When showing continuity by definition, the key is to find a formula for δ. Usually this formula cannot be obtained until enough simplification has been done to $f(x) - f(x_0)$. Therefore it is a good idea to first write done:

> For any given $\varepsilon > 0$, we take $\delta =$ (leave blank for now), then for every $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| = \ldots.$(simplifications).

and then fill in the black for δ when enough simplification is done.

Remark 106. It is important to understand that saying $f(x)$ is continuous at x_0 has two layers of meanings, first, $\lim_{x\to x_0} f(x)$ exists, second, this limit is exactly $f(x_0)$. Correspondingly, $f(x)$ is **not** continuous at x_0[8] if either

- $\lim_{x\to x_0} f(x)$ does not exist, or
- $\lim_{x\to x_0} f(x)$ exists but does not equal $f(x_0)$.

Example 107. Examples of functions that are not continuous.

- We start from the simplest one: $f(x) = \begin{cases} x & x \neq 0 \\ 2 & x = 0 \end{cases}$ is not continuous at 0.
- $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not continuous at $x = 0$.
- The Dirichlet function $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$ is not continuous anywhere.

Theorem 108. *f is continuous at x_0 if and only if for every sequence $x_n \longrightarrow x_0$, $f(x_n) \longrightarrow f(x_0)$.*

Exercise 47. Prove the above theorem.

5.2. Arithmetics of continuous functions.

Theorem 109. *Let f, g be functions continuous at x_0. Then $f \pm g$, $f\,g$ are continuous at x_0. Furthermore if $g(x_0) \neq 0$, f/g is also continuous at x_0.*

Exercise 48. Prove the above theorem.

8. Keep in mind that the best way to understand a definition in mathematical analysis is to construct examples of those that do not satisfy the definition. Once you can construct one example for every single way of possible violation of the requirements of the definition, you can say

Exercise 49. Let $f(x)$ be continuous at x_0, $g(x)$ be not continuous at x_0. Study the continuity of $f \pm g$, $f\,g$, $f/g, g/f$ at x_0. Justify your answers.

Example 110. (Continuity of everyday functions) The following functions are continuous at every $x \in \mathbb{R}$

a) Polynomials $P(x) = a_n x^n + \cdots + a_1 x + a_0$.

b) The exponential function e^x.

c) The functions $\sin x$, $\cos x$.

Proof. The proofs of b), c) involve something beyond 314. So we only prove a) here. Thanks to Theorem 109, all we need to prove is the continuity of $f(x) = x$. Since then the continuity of x^2 follows from $x^2 = x \cdot x$, the continuity of x^3 follows from $x^3 = x^2 \cdot x$, ..., the continuity of x^n follows from $x^n = x^{n-1} \cdot x$. Finally the continuity of the polynomial also follows from Theorem 109.

For any $\varepsilon > 0$, take $\delta = \varepsilon$. Then for all $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| = |x - x_0| < \delta = \varepsilon. \tag{209}$$

The proof ends. □

Example 111. (Rational functions) A rational function is the ratio of two polynomials: $f(x) = \frac{p(x)}{q(x)}$ where p, q are polynomials. Since p, q are continuous at every $x \in \mathbb{R}$, $f(x) = \frac{p(x)}{q(x)}$ is continuous at all $x \in \mathbb{R}$ such that $q(x) \neq 0$.

If $q(x_0) = 0$, then there are two cases.

1. If after cancelling all common factors, $f(x) = \frac{p_1(x)}{q_1(x)}$ with $q_1(x_0) \neq 0$, then $f(x)$ is continuous at x_0.

2. If after cancellation we still have $q_1(x) = 0$, then the limit does not exist.

For example, consider

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 4}. \tag{210}$$

Factorize:

$$x^2 - 3x + 2 = (x - 1)(x - 2); \qquad x^2 - 4 = (x + 2)(x - 2), \tag{211}$$

so

$$f(x) = \frac{x - 1}{x + 2}. \tag{212}$$

By Theorem 96, $f(x)$ is continuous at every $x \in \mathbb{R}$, with $x \neq -2$. At $x = -2$, the limit doesn't exist since $x - 1 \neq 0$.

Exercise 50. Prove that if $q_1(x) = 0$, then the limit does not exist (even if we include $\pm\infty$ as limits).

Example 112. Ratios involving functions other than polynomials are subtle to deal with. For example, although no cancellation can be made for $f(x) = \frac{\sin x}{x}$, it is indeed continuous everywhere, even at $x = 0$.

5.3. Continuity of composite functions.

Definition 113. *Let $f: A \mapsto B$, $g: B \mapsto \mathbb{R}$ where A, B are subsets of $\mathbb{R}$. Then the composition function $g \circ f$ is defined as*

$$(g \circ f)(x) = g(f(x)). \tag{213}$$

Example 114. The function e^{-1/x^2} is a composite function: $e^{-1/x^2} = g \circ f$ with $f(x): \{x \in \mathbb{R}: x \neq 0\} \mapsto \mathbb{R}$ defined as $f(x) = -1/x^2$, and $g(x) = e^x$.

Obviously there are more than one way to write a function into composite functions. For example we also have $e^{-1/x^2} = g_1 \circ f_1$ with $f_1 = 1/x^2$ and $g_1 = e^{-x}$. We can even write it into composition of more than two functions: $e^{-1/x^2} = h_1 \circ h_2 \circ h_3 \circ h_4$ with

$$h_1(x) = e^x, \quad h_2(x) = -x, \quad h_3(x) = x^2, \quad h_4(x) = 1/x. \tag{214}$$

It is important to understand that in general $f \circ g \neq g \circ f$. For example $g = e^x$ and $f = x^2$.

Theorem 115. (Continuity of composite functions) *Let $f: A \mapsto B$, $g: B \mapsto \mathbb{R}$ If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.*

Proof. Given any $\varepsilon > 0$, there is $\delta_1 > 0$ such that for all $|y - f(x_0)| < \delta_1$, $|g(y) - g(f(x_0))| < \varepsilon$; On the other hand, the continuity of f gives: there is $\delta > 0$ such that for all $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \delta_1$. Combine these, we see that $|g(f(x)) - g(f(x_0))| < \varepsilon$. □

Example 116. Show that the following functions are continuous on $\mathbb{R}$.

a) $f_1(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}.$

b) $f_2(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & |x| > 1 \end{cases}.$

Proof. We only prove a). The proof for b) is similar.

Note that $f_1(x) = f \circ g$ with $f = e^x$, $g = -1/x^2$. We know that f is continuous everywhere, and g is continuous everywhere except at $x = 0$. Therefore $f_1(x)$ is continuous at every $x \neq 0$. Since we have already shown in a previous example that $f_1(x)$ is continuous at 0 too, we conclude that $f_1(x)$ is continuous everywhere. □

Remark 117. We will see in future lectures that f_1, f_2 in fact are infinitely differentiable on $\mathbb{R}$, which may be counter-intuitive.

6. Properties of Continuous Functions

Intuitively, a function $f: \mathbb{R} \mapsto \mathbb{R}$ is simply a curve running from far far left to far far right. One naturally expect it to have the following properties:

- Intermediate value. If $f(x_1) = a$ and $f(x_2) = b$, then f takes all intermediate values between a, b;
- Maximum and minimum. Between any $a, b \in \mathbb{R}$, that is over any finite interval, f reaches its maximum and minimum.

Remark 118. It is obvious that one should not expect f to reach maximum or minimum if the interval's size is infinite since the function may be unbounded.

Example 119. One can easily construct examples of discontinuous functions that do not have the above good properties. For example, the function $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leqslant 0 \end{cases}$ does not satisfy the intermediate value property; The function $f(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 1 & |x| > 1 \end{cases}$ does not reach its minimum.

6.1. Intermediate value theorem.

Theorem 120. (Intermediate value) *Let $f(x)$ be continuous on the closed interval $[a, b]$. Then for every $s \in [f(a), f(b)]$ (or $[f(b), f(a)]$ if $f(b) \leqslant f(a)$), there is $\xi \in [a, b]$ such that $f(\xi) = s$.*

Proof. Assume $f(a) \leqslant f(b)$. Let $x_1 = a$, $y_1 = b$. We have $f(x_1) \leqslant s \leqslant f(y_1)$. Define x_n as follows: Suppose x_k, y_k are known. Then

- if $f\left(\frac{x_k + y_k}{2}\right) < s$, set $x_{k+1} = \frac{x_k + y_k}{2}$, $y_{k+1} = y_k$;
- if $f\left(\frac{x_k + y_k}{2}\right) > s$, set $y_{k+1} = \frac{x_k + y_k}{2}$, $x_{k+1} = x_k$;
- if $f\left(\frac{x_k + y_k}{2}\right) = s$, the process terminates and the proof ends.

If the above process never terminates, then we obtain two sequences $\{x_n\}, \{y_n\}$ satisfying $x_{n+1} \geqslant x_n$, $y_{n+1} \leqslant y_n$ and furthermore

$$f(x_n) \leqslant s, \qquad f(y_n) \geqslant s, \qquad |x_n - y_n| \leqslant \frac{|x_{n-1} - y_{n-1}|}{2}. \tag{215}$$

This means $x_n - y_n \longrightarrow 0$.

Now as $\{x_n\}$ is increasing and bounded above, $\lim_{n \longrightarrow \infty} x_n$ exists; On the other hand, since $\{y_n\}$ is decreasing and bounded below, $\lim_{n \rightarrow \infty} y_n$ exists. Because $x_n - y_n \longrightarrow 0$ the two limits are equal. Call it ξ.

Because f is continuous on $[a, b]$, together with comparison theorem we have $\lim_{n \longrightarrow \infty} f(x_n) = f(\xi) \leqslant s$; Similarly we have $\lim_{n \longrightarrow \infty} f(y_n) = f(\xi) \geqslant s$. Combining $f(\xi) \geqslant s$ and $f(\xi) \leqslant s$ we conclude $f(\xi) = s$. □

Remark 121. A few remarks.

- There may be more than one ξ satisfying $f(\xi) = s$. For example consider $f(x) = \sin x$.
- It is not enough to assume $f(x)$ to be continuous over the open interval (a, b). For example, take the Heaviside function $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leqslant 0 \end{cases}$. $f(x)$ is continuous over $(0, 1)$ but it does not take any intermediate value between $f(0) = 0$ and $f(1) = 1$.

Exercise 51. Assume $c < a, d > b$, and $f(x)$ is continuous on (c, d). Prove that $f(x)$ is continuous on $[a, b]$, that is For every $x_0 \in [a, b]$, we have $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall x \in [a, b]$ $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon$. Or in other words:

1. $\forall x_0 \in (a, b)$, $\lim_{x \to x_0} f(x) = f(x_0)$;
2. $\lim_{x \to a+} f(x) = f(a)$; $\lim_{x \to b-} f(x) = f(b)$.

Example 122. One application of the intermediate value theorem is to find zeroes of functions. For example, one can prove:

Theorem. Any odd-degree polynomial has at least one real root.

Proof. Denote the polynomial as

$$P(x) = a_{2k-1} x^{2k-1} + a_{2k-2} x^{2k-2} + \cdots + a_1 x + a_0, \qquad a_{2k-1} \neq 0 \tag{216}$$

By the Intermediate Value Theorem 107 all we need to do is to find $a, b \in \mathbb{R}$ such that $f(a) < 0$, $f(b) > 0$. Write

$$P(x) = a_{2k-1} x^{2k-1} \left[1 + \frac{a_{2k-2}}{a_{2k-1}} \frac{1}{x} + \frac{a_{2k-3}}{a_{2k-1}} \frac{1}{x^2} + \cdots + \frac{a_1}{a_{2k-1}} \frac{1}{x^{2k-2}} + \frac{a_0}{a_{2k-1}} \frac{1}{x^{2k-1}} \right]. \tag{217}$$

Then we can prove that[9]

$$1 + \frac{a_{2k-2}}{a_{2k-1}} \frac{1}{x} + \frac{a_{2k-3}}{a_{2k-1}} \frac{1}{x^2} + \cdots + \frac{a_1}{a_{2k-1}} \frac{1}{x^{2k-2}} + \frac{a_0}{a_{2k-1}} \frac{1}{x^{2k-1}} \longrightarrow 1 \tag{218}$$

as $x \longrightarrow \infty$ or $x \longrightarrow -\infty$. Therefore there is $M > 0$ such that for all $|x| > M$, the sign of $P(x)$ is the same as that of $a_{2k-1} x^{2k-1}$. Thus there are $a, b \in \mathbb{R}$ such that $f(a) < 0$, $f(b) > 0$.

6.2. Maximum and Minimum.

To show the maximum and minimum property, we first need the following boundedness result.

Theorem 123. (Boundedness) *Let $f(x)$ be continuous on $[a, b]$ for $a, b \in \mathbb{R}$, then $f(x)$ is bounded on $[a, b]$. That is there is $M \in \mathbb{R}$ such that $|f(x)| \leqslant M$ for all $x \in [a, b]$.*

Proof. We prove by contradiction. Assume that

$$\forall M \in \mathbb{R}\ \exists x \in [a, b], \qquad |f(x)| > M. \tag{219}$$

Taking $M = 1, 2, 3, \ldots$ we obtain a sequence $\{x_n\}$ satisfying $|f(x_n)| > n$. Now apply Bolzano-Weierstrass to $\{x_n\}$ we obtain a converging subsequence $x_{n_k} \longrightarrow \xi$. As f is continuous, we conclude $f(x_{n_k})$ is a bounded sequence; But at the same time $|f(x_{n_k})| > n_k$ implies $f(x_{n_k})$ is an unbounded sequence. Contradiction. □

Exercise 52. Does the theorem still hold if the closed interval $[a, b]$ is replaced by the open interval (a, b) or the half-open-half-closed ones $(a, b]$ or $[a, b)$? Justify your answers.

Theorem 124. (Maximum and minimum) *Let $f(x)$ be continuous on $[a, b]$ for $a, b \in \mathbb{R}$. Then f reaches both maximum and minimum, that is there are $x_{\max}, x_{\min} \in [a, b]$ such that*

$$f(x_{\min}) \leqslant f(x) \leqslant f(x_{\max}) \qquad \forall x \in [a, b]. \tag{220}$$

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9. If it's in an exam, you have to prove it in detail.

Proof. From the boundedness theorem 110 we know that there is M such that $|f(x)| \leqslant M$ for all $x \in [a, b]$. Let $s = \sup_{x \in [a,b]} f(x) := \sup \{f(x) : x \in [a, b]\}$. Then $s \in \mathbb{R}$ and there is a sequence x_n such that $f(x_n) \longrightarrow s$. As $x_n \in [a, b]$ the sequence is bounded so we can apply Bolzano-Weierstrass to obtain a converging subsequence $x_{n_k} \longrightarrow x_{\max} \in [a, b]$. The continuity of f at ξ now gives

$$f(x_{\max}) = \lim_{k \longrightarrow \infty} f(x_{n_k}) = s. \tag{221}$$

That is f attains maximum at $x_{\max}$. The other half of the theorem can be proved similarly. □

6.3. Continuity of Inverse Functions.

Theorem 125. *A continuous real function $f(x): A \mapsto B := f(A)$ has an inverse if and only if $f(x)$ is strictly increasing, that is $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, or strictly decreasing, that is $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.*

Proof.

- "If". If $f(x)$ is strictly increasing, then f is one-to-one. As $B = f(A)$ f is also onto. So there is an inverse. Similar for strictly decreasing.
- "only if". We try to show that if f is one-to-one then it has to be either strictly increasing or strictly decreasing. Take any $x_1 < x_2$. There are two cases:
  - ○ $f(x_1) < f(x_2)$. We show that f must be strictly increasing.
    1. First show for all $x > x_2$, we have $f(x) > f(x_2)$. Assume the contrary. Then either there is $x > x_2$ such that $f(x) = f(x_2)$, which already leads to contradiction to the fact that f is one-to-one, or there is $x > x_2$ such that $f(x) < f(x_2)$. Now compare $f(x)$ and $f(x_1)$. If $f(x) \leqslant f(x_1)$, then by intermediate value theorem there is $\xi \in [x_2, x]$ (thus $\xi \neq x_1$) with $f(\xi) = f(x_1)$, contradiction. If $f(x) > f(x_1)$, then again by intermediate value theorem there is $\xi \in [x_1, x_2]$ such that $f(\xi) = f(x)$. Contradiction.
    2. Similarly one can show that for all $x < x_1$, $f(x) < f(x_1)$.
    3. Next we show that for all $x \in (x_1, x_2)$, $f(x_1) < f(x) < f(x_2)$. Assume not, then either there is x with $f(x) \geqslant f(x_2)$ or there is x with $f(x) \leqslant f(x_1)$. Either case, we can apply intermediate value theorem to reach a contradiction.
    4. Now we show that for all $x_3 < x_4$, $f(x_3) < f(x_4)$. There are several cases.
       - a. $x_3 \in$ one of $(-\infty, x_1), [x_1, x_2], (x_2, \infty)$ while $x_4 \in$ another. Then from what we have shown $f(x_3) < f(x_4)$.
       - b. $x_3, x_4 \in$ the same one of $(-\infty, x_1), [x_1, x_2], (x_2, \infty)$. We prove for the case $x_3, x_4 \in (-\infty, x_1)$, other cases are similar. Assume $f(x_3) < f(x_4)$ does not hold. If $f(x_3) = f(x_4)$, we already have contradiction; If $f(x_3) > f(x_4)$, then we have $f(x_3) > f(x_4) < f(x_1)$, application of intermediate value theorem leads to contradiction.
  - ○ $f(x_1) > f(x_2)$. Similarly we can show that in this case f must be strictly decreasing. □

Exercise 53. Prove in detail the case $f(x_1) > f(x_2)$.

Remark 126. The above proof is kind of messy. The idea is however simple: By intermediately value theorem, whenever we have $x_1 < x_2 < x_3$ with $f(x_1) \leqslant f(x_2) \geqslant f(x_3)$ or $f(x_1) \geqslant f(x_2) \leqslant f(x_3)$ then f cannot be one-to-one. However I haven't been able to find a simple proof yet.

Theorem 127. *Let $f: A \mapsto B$ be continuous, onto, and strictly increasing (strictly decreasing). Then the inverse $g: B \mapsto A$ is continuous, onto, and strictly increasing (strictly decreasing).*

Proof. That g is onto is clear. To see that g is strictly increasing, take $y_1 < y_2$. Then we have

$$f(g(y_1)) = y_1 < y_2 = f(g(y_2)). \tag{222}$$

As f is strictly increasing, we must have $g(y_1) < g(y_2)$ since otherwise we would have $f(g(y_1)) \geqslant f(g(y_2))$.

Now we show that g is continuous. Assume the contrary, that is assume g is not continuous at some y_0. Then by definition there is $\varepsilon_0 > 0$ such that there is $y_n \longrightarrow y_0$ but $|g(y_n) - g(y_0)| > \varepsilon_0$. Now let

$$\delta_0 = \min \{f(g(y_0)) - f(g(y_0) - \varepsilon_0), f(g(y_0) + \varepsilon_0) - f(g(y_0))\} > 0, \tag{223}$$

Note that $\delta_0 > 0$ is because f is strictly increasing, we obtain

$$|y_{n_k} - y_0| = |f(g(y_{n_k})) - f(g(y_0))| > \delta_0 > 0. \tag{224}$$

But this contradicts $y_{n_k} \longrightarrow y_0$.

Thus ends the proof. □

Example 128. (Log) We see that $\ln x$ is continuous for $x > 0$.