Differentiation

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1. DERIVATIVES

1.1. Definition.

Definition 1. *Let f be a real function. At a point x_0 inside its domain, if the limit*

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{1}$$

exists and is finite, we say f is differentiable at x_0, and call the limit its derivative at x_0, denoted $f'(x_0)$. If the limit does not exist, we say f is not differentiable at x_0. If f is differentiable at all $x \in E$ where $E \subseteq \mathbb{R}$, we say f is differentiable on E. If f is differentiable at every point of its domain, we say f is differentiable.

Remark 2. Equivalently, one can define differentiability through the limit

$$\lim_{h \longrightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}. \tag{2}$$

That is f is differentiable at x_0 if the above limit exists.

Exercise 1. Is the following an equivalent definition of differentiability? Justify your answer.

Let f be a real function. At a point x_0 inside its domain, if the limit

$$\lim_{h \longrightarrow 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} \tag{3}$$

exists, we say f is differentiable at x_0.

Exercise 2. Let $f(x) = a$ for all x in its domain. Prove that $f'(x) = 0$. Note that you have to establish differentiability of $f(x)$.

Remark 3. Recall that in the definition of limits, we require $0 < |x - x_0|$. This is crucial in the limit (3) since at $x = x_0$ we have $\frac{0}{0}$.

Example 4. Let $f(x) = x$. Prove that $f(x)$ is a differentiable function.
Solution. We need to prove that f is differentiable at every $x_0 \in \mathbb{R}$.

For every $x_0 \in \mathbb{R}$,

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \longrightarrow x_0} \frac{x - x_0}{x - x_0} = \lim_{x \longrightarrow x_0} 1 = 1. \tag{4}$$

So $(x^1)' = 1$.

Exercise 3. Let $f(x) = 1$. Prove it is a differentiable function.

Example 5. Let $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Prove that $f(x)$ is differentiable at $x = 0$ and find $f'(0)$.

Solution. By definition we check the limit

$$\lim_{x \longrightarrow 0} \frac{x^2 \sin \frac{1}{x} - 0}{x} = \lim_{x \to 0} \left(x \sin \frac{1}{x} \right). \tag{5}$$

Since

$$-|x| \leqslant x\sin\frac{1}{x} \leqslant |x| \tag{6}$$

and

$$\lim_{x\to 0}(-|x|) = 0 = \lim_{x\to 0}|x|, \tag{7}$$

by Squeeze Theorem we have

$$\lim_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0. \tag{8}$$

Therefore $f(x)$ is differentiable at $x=0$ and $f'(0)=0$.

Example 6. Let $f(x) = \begin{cases} x\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$. Prove that $f(x)$ is not differentiable at $x=0$.

Solution. By definition we need to show the limit

$$\lim_{x\longrightarrow 0}\frac{x\sin\frac{1}{x}-0}{x} = \lim_{x\to 0}\left(\sin\frac{1}{x}\right) \tag{9}$$

does not exist.

Take $x_n = \frac{1}{n\pi}$, $y_n = \frac{1}{(2n+1/2)\pi}$. Then we have

$$\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n; \qquad \forall n, x_n \neq 0, y_n \neq 0; \tag{10}$$

$$\lim_{n\to\infty}\sin\left(\frac{1}{x_n}\right) = \lim_{n\to\infty}\sin(n\pi) = \lim_{n\to\infty} 0 = 0, \tag{11}$$

$$\lim_{n\to\infty}\sin\left(\frac{1}{y_n}\right) = \lim_{n\to\infty}\sin((2n+1/2)\pi) = \lim_{n\to\infty} 1 = 1. \tag{12}$$

As $1\neq 0$, $\lim_{x\to 0}\left(\sin\frac{1}{x}\right)$ does not exist.

1.2. Arithmetics.

Theorem 7. (Arithmetics of derivatives) *Let f, g be differentiable at x_0. Then*

a) $f\pm g$ is differentiable at x_0 with $(f\pm g)'(x_0) = f'(x_0)\pm g'(x_0)$.

b) **(Leibniz rule)** *fg is differentiable at x_0 with $(fg)'(x_0) = f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0)$.*

c) If $g(x_0)\neq 0$, then f/g is differentiable at x_0 with

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\,g(x_0) - f(x_0)\,g'(x_0)}{g(x_0)^2}. \tag{13}$$

Proof.

a) We have

$$\frac{(f+g)(x)-(f+g)(x_0)}{x-x_0} = \frac{f(x)-f(x_0)}{x-x_0} + \frac{g(x)-g(x_0)}{x-x_0}. \tag{14}$$

Since

$$\lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}+\frac{g(x)-g(x_0)}{x-x_0}\right]=f'(x_0)+g'(x_0) \tag{15}$$

The limit

$$\frac{(f+g)(x)-(f+g)(x_0)}{x-x_0} \tag{16}$$

also exists and equals $f'(x_0)+g'(x_0)$. The case $f-g$ can be proved similarly.

b) We have

$$\frac{(fg)(x)-(fg)(x_0)}{x-x_0}=f(x)\,\frac{g(x)-g(x_0)}{x-x_0}+g(x_0)\,\frac{f(x)-f(x_0)}{x-x_0}. \tag{17}$$

Since

$$\lim_{x\to x_0} f(x)=f(x_0);\quad \lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}=g'(x_0);\quad \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0) \tag{18}$$

we reach

$$\lim_{x\to x_0}\frac{(fg)(x)-(fg)(x_0)}{x-x_0}=f(x_0)\,g'(x_0)+f'(x_0)\,g(x_0). \tag{19}$$

c) We only prove the last one. In light of b), it suffices to prove

$$\left(\frac{1}{g}\right)'=-\frac{g'(x_0)}{g^2(x_0)}. \tag{20}$$

Write

$$\frac{\frac{1}{g(x)}-\frac{1}{g(x_0)}}{x-x_0}=-\frac{\frac{g(x)-g(x_0)}{x-x_0}}{g(x)\,g(x_0)}. \tag{21}$$

Note that both the denominator and the numerator have limits, and furthermore the limit of the denominator is not 0. So we have the limit of the ratio exists and

$$\lim_{x\longrightarrow x_0}\left[-\frac{\frac{g(x)-g(x_0)}{x-x_0}}{g(x)\,g(x_0)}\right]=-\frac{\lim_{x\longrightarrow x_0}\frac{g(x)-g(x_0)}{x-x_0}}{\lim_{x\longrightarrow x_0}g(x)\,g(x_0)}=-\frac{g'(x_0)}{g(x_0)^2}. \tag{22}$$

Thus ends the proof. □

Example 8. Let $n\in\mathbb{N}$. Prove that x^n is differentiable everywhere for $n>0$ and differentiable at $x\neq 0$ for all $n<0$.

Solution.

- Let $P(n)$ be "x^n is differentiable everywhere". We prove $P(n)$ is true for all $n\in\mathbb{N}$ through induction.
  - Base: $P(1)=$"x is differentiable everywhere" has already been proved above.
  - $P(n)\Longrightarrow P(n+1)$. Assume x^n is differentiable everywhere. Let $x_0\in\mathbb{R}$ be arbitrary. Then since both x^n and x are differentiable at x_0, we have

$$x^{n+1}=(x^n)\cdot x \tag{23}$$

is differentiable at x_0.

- Let $P(n)$ be "x^{-n} is differentiable at every $x_0 \neq 0$". We prove $P(n)$ is true for all $n \in \mathbb{N}$ through induction.
  - Base: $P(1) =$ "$1/x$ is differentiable at every $x_0 \neq 0$." We have shown that x is differentiable everywhere. Let $x_0 \neq 0$. We check

$$\lim_{x \to x_0} \frac{1-1}{x - x_0} = 0 \tag{24}$$

    therefore 1 is differentiable at x_0. By the above theorem we have $1/x$ differentiable at x_0.
  - $P(n) \Longrightarrow P(n+1)$: Left as exercise.

Theorem 9. *The following functions are differentiable everywhere.*

$$\sin x, \qquad \cos x, \qquad e^x, \qquad \textit{Polynomials.} \tag{25}$$

1.3. Composite and inverse functions.

Theorem 10. (Chain rule) *If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then the composite function $g \circ f$ is differentiable at x_0 and satisfy*

$$(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0). \tag{26}$$

Proof. Set

$$h(y) := \begin{cases} \frac{g(y) - g(f(x_0))}{y - f(x_0)} & y \neq f(x_0) \\ g'(f(x_0)) & y = f(x_0) \end{cases}. \tag{27}$$

Then we have $h(y)$ satisfying $\lim_{y \longrightarrow f(x_0)} h(y) = h(f(x_0))$.

Now write

$$\frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = h(f(x))\, \frac{f(x) - f(x_0)}{x - x_0}. \tag{28}$$

By Lemma 13 we have $\lim_{x \longrightarrow x_0} f(x) = f(x_0)$. Thus taking limit of both sides of (28) we reach

$$\lim_{x \longrightarrow x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = \left(\lim_{x \longrightarrow x_0} h(f(x)) \right) \left(\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \right) = h(f(x_0))\, f'(x_0) \tag{29}$$

and the proof ends. □

Remark 11. Naturally one may want to prove through

$$\frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)}\, \frac{f(x) - f(x_0)}{x - x_0} \tag{30}$$

and try to show

$$\lim_{x \longrightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} = g'(f(x_0)). \tag{31}$$

However this does not work because it may happen that $f(x) - f(x_0) = 0$. The above trick overcomes this difficulty.

Theorem 12. **(Derivative of inverse function)** *Let f be differentiable at x_0 with $f'(x_0) \neq 0$. Then if f has an inverse function g, then g is differentiable at $y_0 = f(x_0)$ and satisfies $g'(f(x_0)) = 1/f'(x_0)$ or equivalently $g'(y_0) = 1/f'(g(y_0))$.*

Proof. Since f has an inverse function, f is either strictly increasing or strictly decreasing. Furthermore g is continuous, and also strictly increasing or decreasing.

Let $y_0 = f(x_0)$. We compute

$$\frac{g(y) - g(y_0)}{y - y_0} = \frac{g(y) - g(y_0)}{f(g(y)) - f(g(y_0))} = \left(\frac{f(g(y)) - f(g(y_0))}{g(y) - g(y_0)} \right)^{-1}. \tag{32}$$

Note that as f, g are both strictly increasing/decreasing, all the denominators in the above formula are nonzero. To show that the limit exists, we recall that $\lim F(x)$ exists at x_0 if for all $x_n \longrightarrow x_0$ the limit of $F(x_n)$ exists.

Take $y_n \longrightarrow y_0$. By continuity of g we have $g(y_n) \longrightarrow g(y_0)$. The differentiability of f at $g(y_0)$, that is the existence of the limit $\lim_{x \longrightarrow g(y_0)} \frac{f(x) - f(g(y_0))}{x - g(y_0)}$, then gives

$$\lim_{n \longrightarrow \infty} \frac{f(g(y_n)) - f(g(y_0))}{g(y_n) - g(y_0)} = f'(g(y_0)) = f'(x_0) \neq 0. \tag{33}$$

Thus ends the proof. □

Example 13. Assume that we are given $\tan'(x) = \frac{1}{\cos^2 x}$, find $\arctan'$.
Solution. We have

$$\arctan'(y) = \frac{1}{\tan'(x)} = \cos^2(x). \tag{34}$$

What we need now is to represent $\cos^2(x)$ by $y = \tan x$. It is clear that $\cos^2 x = \frac{1}{1 + y^2}$ so $\arctan'(y) = \frac{1}{1 + y^2}$.

Example 14. Assume that we are given $(e^x)' = e^x$. Find $(\ln x)'$.
Solution. We have

$$(\ln)'(y) = \frac{1}{(e^x)'} = \frac{1}{e^x} = \frac{1}{y} \tag{35}$$

since $y = e^x$.

Example 15. **($f'(x_0) = 0$)** Consider $f(x) = x^3$. Then $g(y) = y^{1/3}$. We see that at $x_0 = 0$, g is not differentiable.

Exercise 4. Prove the following "Toy L'Hospital's Rule".

Let f, g be differentiable at x_0, and furthermore $f(x_0) = g(x_0) = 0$. Then if $g'(x_0) \neq 0$, we have

$$\lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}. \tag{36}$$

Then apply it to evaluate the following limits:

$$\lim_{x \longrightarrow 1} \frac{x^2 - 1}{x - 1}; \qquad \lim_{x \longrightarrow 0} \frac{\sin x}{x}; \qquad \lim_{x \longrightarrow 0} \frac{1 - \cos x}{x^2}. \tag{37}$$

Do you encounter any difficulty?

2. Properties of Differentiable Functions

2.1. Differentiability and continuity.

Lemma 16. (Differentiable functions are continuous) *If $f(x)$ is differentiable at x_0, then $f(x)$ is continuous at x_0.*

Proof. Since $f(x)$ is differentiable at x_0, we have by definition

$$\lim_{x\longrightarrow x_0} \frac{f(x)-f(x_0)}{x-x_0} = L\in\mathbb{R} \tag{38}$$

Now write

$$f(x) = f(x_0) + (x-x_0)\,\frac{f(x)-f(x_0)}{x-x_0} \tag{39}$$

and take limit $x\longrightarrow x_0$, we have

$$\lim_{x\to x_0} f(x) = f(x_0) + \left[\lim_{x\to x_0} x-x_0\right]\left[\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\right] = f(x_0) + 0\cdot L = f(x_0) \tag{40}$$

Therefore $f(x)$ is continuous at x_0. □

Remark 17. One can also prove using definition as follows. Since $f(x)$ is differentiable at x_0, we have by definition

$$\lim_{x\longrightarrow x_0} \frac{f(x)-f(x_0)}{x-x_0} = L\in\mathbb{R} \tag{41}$$

Take $\delta_1>0$ such that for all $0<|x-x_0|<\delta_1$,

$$\left|\frac{f(x)-f(x_0)}{x-x_0} - L\right| < 1 \Longrightarrow |f(x)-f(x_0)| < (|L|+1)\,|x-x_0|. \tag{42}$$

Now for any $\varepsilon>0$, take $\delta=\min\left\{\delta_1, \frac{\varepsilon}{|L|+1}\right\}$. We have, for all $0<|x-x_0|<\delta$,

$$|f(x)-f(x_0)| < (|L|+1)\,|x-x_0| < (|L|+1)\,\delta \leqslant \varepsilon. \tag{43}$$

Exercise 5. Prove or disprove the converse claim:

If $f(x)$ is continuous at x_0, then it is differentiable at x_0.

2.2. Maximum and minimum.

Definition 18. (Local maximum/minimum) *Let $f\colon[a,b]\mapsto\mathbb{R}$ be a real function. We say f has a local maximum at $x_0\in(a,b)$ if there exists some $\delta>0$ such that $f(x)\leqslant f(x_0)$ for all $x\in(x_0-\delta, x_0+\delta)$. This x_0 is said to be a local maximizer. We say f has a local minimum at x_0 if there exists some $\delta>0$ such that $f(x)\geqslant f(x_0)$ for all $x\in(x_0-\delta,x_0+\delta)$. This x_0 is said to be a local minimizer.*

Exercise 6. Find all local maxima/minima for the following functions.

a) $f(x)=1$.
b) $f(x)=x^2$;
c) $f(x)=\sin x$;
d) $f(x)=x\sin x$;
e) $f(x)=\sin(1/x)$.

Theorem 19. *If f is differentiable at its local maximizer or minimizer, then the derivative is 0 there.*

Proof. Assume x_0 is a local maximizer. Take $x_n\in(x_0,x_0+\delta)$ with $\lim_{n\longrightarrow\infty}x_n=x_0$. Since f is differentiable at x_0, we have

$$f'(x_0) = \lim_{x\longrightarrow x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{n\longrightarrow\infty}\frac{f(x_n)-f(x_0)}{x-x_0}. \tag{44}$$

But as $f(x_n) - f(x_0) \leqslant 0$ for all n, by comparison theorem we reach $f'(x_0) \leqslant 0$.

Now take $x_n \in (x_0 - \delta, x_0)$ with $\lim_{n\to\infty} x_n = x_0$. Similar argument as above gives $f'(x_0) \geqslant 0$. Therefore $f'(x_0) = 0$.

The proof for the local minimizer case is similar and left as exercise. □

Remark 20. It may happen that f is not differentiable at its maximizer or minimizer. For example $f(x) = |x|$.

Example 21. Consider $f(x) = x \sin(1/x)$. Then its local maximizers and minimizers can be obtained by solving

$$0 = f'(x) = \sin(1/x) - \frac{x}{x^2} \cos(1/x) \Longrightarrow \tan(1/x) = 1/x. \tag{45}$$

The solutions have to be obtained numerically as it is not possible to represent them using elementary functions.

Note. It is important to note that the local maximizers and local minimizers of $f(x)$ are not $x_n = \frac{1}{(2n + 1/2)\pi}$ and $y_n = \frac{1}{(2n - 1/2)\pi}$!

2.3. Mean value theorem.

Theorem 22. (Rolle's Theorem) *Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there is $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

Remark 23. Before proving the theorem, we illustrate the necessity of the assumptions.

- f is continuous on $[a, b]$. If not, $f(x) = \begin{cases} x & 0 \leqslant x < 1 \\ 0 & x = 1 \end{cases}$.
- f is differentiable on (a, b). If not, $f(x) = |x|$ over $[-1, 1]$.

Proof. Since f is continuous on $[a, b]$, there are $x_{\min}, x_{\max} \in [a, b]$ such that $f(x_{\min})$ is the minimum and $f(x_{\max})$ is the maximum. If one of them is different from a, b, then $f' = 0$ there due to Theorem 16. Otherwise we have $f(a) = f(b) = f(x_{\min}) = f(x_{\max}) \Longrightarrow f(x)$ is constant on $[a, b]$, consequently $f'(x) = 0$ for all $x \in (a, b)$. □

Exercise 7. (Rolle over $\mathbb{R}$) Let f be continuous and differentiable on $\mathbb{R}$. If $\lim_{x\longrightarrow+\infty} f(x) = \lim_{x\longrightarrow-\infty} f(x)$, then there is $\xi \in \mathbb{R}$ such that $f'(\xi) = 0$.

Theorem 24. (Mean Value Theorem) *Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $\xi \in (a, b)$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}. \tag{46}$$

Proof. Set $g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$ and apply Rolle's Theorem. □

Remark 25. When the interval has infinite size, the Mean Value Theorem may not hold (even if we accept $(f(b) - f(a))/\infty = 0$). An example is $f(x) = \arctan x$.

Theorem 26. (Cauchy's extended mean value theorem) *Let f, g be continuous over $[a, b]$ and differentiable over (a, b). Further assume $g(a) \neq g(b)$. Then there is $\xi \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}. \tag{47}$$

Exercise 8. Prove Cauchy's MVT. (Hint: take $h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g(x)$ and apply MVT).

3. Monotonicity and L'Hospital

3.1. Monotonicity.

Theorem 27. *Let f be defined over $[a,b]\subseteq\mathbb{R}$. Here a,b can be extended real numbers. Suppose f is continuous on $[a,b]$ and differentiable on (a,b). Then*

a) *f is increasing if and only if $f'(x)\geqslant 0$ for all $x\in(a,b)$; f is decreasing if and only if $f'(x)\leqslant 0$ for all $x\in(a,b)$.*

b) *f is strictly increasing if $f'(x)>0$ for all $x\in(a,b)$; f is strictly decreasing if $f'(x)<0$ for all $x\in(a,b)$.*

c) *f is a constant if and only if $f'(x)=0$ for all $x\in(a,b)$.*

Proof.

a) We prove the increasing case here.

Let f be increasing, we show $f'(x)\geqslant 0$. Take any $x_0\in(a,\, b)$. Since f is increasing, $f(x)\geqslant f(x_0)$ when $x>x_0$ and $f(x)\leqslant f(x_0)$ when $x<x_0$, thus

$$\frac{f(x)-f(x_0)}{x-x_0}\geqslant 0 \tag{48}$$

for all $x\neq x_0$. As f is differentiable at x_0, taking limit of both sides leads to $f'(x_0)\geqslant 0$.

Let $f'(x)\geqslant 0$ for all $x\in(a,b)$. Assume f is not increasing. Then there are $x_1<x_2$ such that $f(x_1)>f(x_2)$. Apply Mean Value Theorem we have there must exist $\xi\in(x_1,x_2)\subseteq(a,b)$ such that

$$f'(\xi)=\frac{f(x_1)-f(x_2)}{x_1-x_2}<0. \tag{49}$$

Contradiction.

b) The proof is similar to the corresponding part of a).

c) The proof is left as exercise. □

Remark 28. Note that $f(x)$ strictly increasing $\not\Longrightarrow$ $f'(x)>0$ everywhere. An examples is $f(x)=x^3$.

Example 29. Prove that $e^x>1+x$ for all $x>0$.

Proof. Let $f(x)=e^x-1-x$. We see that $f(0)=0$. To show $f(x)>0$ it suffices to show f is strictly increasing. Calculate

$$f'(x)=e^x-1>0 \tag{50}$$

for all $x>0$. Therefore f is strictly increasing and consequently $f(x)>0$ for all $x>0$. □

Example 30. Prove

$$\frac{x}{1+x}\leqslant\ln(1+x)\leqslant x \tag{51}$$

for all $x>-1$.

Proof. For the first inequality let $f(x)=\ln(1+x)-\frac{x}{1+x}$. We have $f(0)=0$ so all we need to show is $f(x)\geqslant f(0)$. Calculate

$$f'(x)=\frac{x}{(1+x)^2}. \tag{52}$$

Thus $f(x) \geqslant 0$ when $x > 0$ and $f(x) \leqslant 0$ when $x < 0$. Consequently $f(x) \geqslant f(0)$.

For the second inequality let $g(x) = x - \ln(1+x)$. We have $g(0) = 0$ and need to show $g(x) \geqslant g(0)$ for all x. Calculate

$$g'(x) = \frac{x}{1+x}. \tag{53}$$

For $x > -1$ we have $g'(x) > 0$ if $x > 0$ and <0 if $x < 0$. □

Example 31. Prove

$$\arctan\frac{1+x}{1-x} = \arctan x + \frac{\pi}{4} \tag{54}$$

for $-1 < x < 1$.

Proof. Set $x = 0$ we have

$$\arctan\frac{1+0}{1-0} = \arctan 0 + \frac{\pi}{4}. \tag{55}$$

Therefore all we need to show is

$$h(x) := \arctan\frac{1+x}{1-x} - \arctan x \tag{56}$$

is a constant for $-1 < x < 1$. Once this is shown, we have $h(x) = h(0) = \frac{\pi}{4}$.

Taking derivative, we have

$$h'(x) = \frac{\left(\frac{1+x}{1-x}\right)'}{1+\left(\frac{1+x}{1-x}\right)^2} - \frac{1}{1+x^2} = \frac{\frac{1\cdot(1-x)-(-1)\cdot(1+x)}{(1-x)^2}}{\frac{(1-x)^2+(1+x)^2}{(1-x)^2}} - \frac{1}{1+x^2} = 0. \tag{57}$$

Thus ends the proof. □

3.2. L'Hospital's Rule.

Theorem 32. (L'Hospital's Rule) *Let $x_0 \in (a,b)$ and $f(x), g(x)$ be differentiable on $(a,b) - \{x_0\}$. Assume that $\lim_{x\longrightarrow x_0} f(x) = \lim_{x\longrightarrow x_0} g(x) = 0$. Then if $\lim_{x\longrightarrow x_0} \frac{f'(x)}{g'(x)}$ exists and $g'(x) \neq 0$ for $x \in (a,b) - \{x_0\}$, the following holds.*

$$\lim_{x\longrightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x\longrightarrow x_0} \frac{f'(x)}{g'(x)}. \tag{58}$$

Remark 33. Note that there are four conditions for the application of L'Hospital's rule:

1. $f(x), g(x)$ are differentiable on $(a,b) - \{x_0\}$;
2. $\lim_{x\longrightarrow x_0} f(x) = \lim_{x\longrightarrow x_0} g(x) = 0$;
3. $\lim_{x\longrightarrow x_0} \frac{f'(x)}{g'(x)}$ exists;
4. $g'(x) \neq 0$ for $x \in (a,b) - \{x_0\}$.

Proof. (of L'Hospital's Rule) Since $\lim_{x\longrightarrow x_0} f(x) = \lim_{x\longrightarrow x_0} g(x) = 0$ we can define $f(x_0) = g(x_0) = 0$. After such definition f, g becomes continuous over (a,b). Now for any $x \in (a,b)$, we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)} \tag{59}$$

for some ξ between x, x_0, thanks to Cauchy's MVT. Now taking limit $x \longrightarrow x_0$, we have $\xi \longrightarrow x_0$ and the conclusion follows. $\square$

3.3. Applications of L'Hospital's rule.

Example 34. Find $\lim_{x\longrightarrow 0} \frac{x \sin x}{x^2}$.

We see that the conditions for L'Hospital's rule is satisfied. Therefore

$$\lim_{x\longrightarrow 0} \frac{x \sin x}{x^2} = \lim_{x\longrightarrow 0} \frac{\sin x + x \cos x}{2\,x} \tag{60}$$

if the latter exists. Now this second limit still satisfies the conditions for L'Hospital and consequently we have

$$\lim_{x\longrightarrow 0} \frac{\sin x + x \cos x}{2\,x} = \lim_{x\longrightarrow 0} \frac{2 \cos x - x \sin x}{2}. \tag{61}$$

But this last limit exists and equals 1. Therefore

$$\lim_{x\longrightarrow 0} \frac{x \sin x}{x^2} = 1. \tag{62}$$

Remark 35. L'Hospital's rule still holds when $x_0 = \pm\infty$, $\lim_{x\rightarrow x_0} \frac{f'(x)}{g'(x)} = \pm\infty$, or $\lim_{x\longrightarrow x_0} f$, $\lim_{x\longrightarrow x_0} g = \pm\infty$. We will prove for the following situation and leave other cases as exercise.

$$x_0 = +\infty; \qquad \lim_{x\longrightarrow\infty} f(x) = \lim_{x\longrightarrow\infty} g(x) = +\infty; \qquad \lim_{x\longrightarrow x_0} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}.$$

Proof. Let $\varepsilon > 0$ be arbitrary. We try to find $M > 0$ such that

$$\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon \qquad \forall x > M. \tag{63}$$

Set any positive number $\delta < \min\left\{\frac{1}{2}, \frac{\varepsilon}{1 + 4\,(|L| + 1)}\right\}$.

Take $M_1 > 0$ such that

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \delta \qquad \forall x > M_1. \tag{64}$$

Take $x_0 = M_1 + 1$. Fix x_0 now. Then take $M > 0$ such that

$$|f(x_0)| < \delta\, f(x); \qquad |g(x_0)| < \delta\, g(x) \qquad \forall x > M. \tag{65}$$

Now for all $x > M$, we have

$$\frac{1-\delta}{1+\delta} \frac{f(x)}{g(x)} < \frac{f(x) - f(x_0)}{g(x) - g(x_0)} < \frac{1+\delta}{1-\delta} \frac{f(x)}{g(x)} \tag{66}$$

which gives

$$\frac{1-\delta}{1+\delta} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} < \frac{f(x)}{g(x)} < \frac{1+\delta}{1-\delta} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} \tag{67}$$

Now we apply Cauchy's MVT to obtain

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)} \tag{68}$$

for some $\xi \in (x_0, x)$. Recalling our choice of x_0, we conclude

$$\left| \frac{f(x) - f(x_0)}{g(x) - g(x_0)} - L \right| < \delta. \tag{69}$$

This gives

$$\left| \frac{1+\delta}{1-\delta} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} - L \right| \leqslant \left| \frac{f(x) - f(x_0)}{g(x) - g(x_0)} - L \right| + \left| \frac{2\,\delta}{1-\delta} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} \right| < \delta + \frac{2\,\delta}{1-\delta} (|L| + \delta) \tag{70}$$

and

$$\left| \frac{1-\delta}{1+\delta} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} - L \right| < \delta + \frac{2\,\delta}{1+\delta} (|L| + \delta). \tag{71}$$

Recall that $0 < \delta < \frac{1}{2}$ and $\delta < \frac{\varepsilon}{1 + 4\,(|L| + 1)}$, we have

$$\delta + \frac{2\,\delta}{1-\delta} (|L| + \delta) < \delta + 4\,\delta\,(|L| + 1) < \varepsilon, \;\; \delta + \frac{2\,\delta}{1+\delta} (|L| + \delta) < \delta + 2\,\delta\,(|L| + 1) < \varepsilon. \tag{72}$$

Therefore

$$\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon \qquad \forall x > M. \tag{73}$$

Thus ends the proof. □

Exercise 9. Let $a < b < c$ be real numbers. Let $L \in \mathbb{R}$. Prove that

$$|b - L| < \max\{|a - L|, |c - L|\}. \tag{74}$$

Exercise 10. Is it possible to prove the above situation through the following:

Set $F(z) := (f(x(z)))^{-1}$, $G(z) := (g(x(z)))^{-1}$ where $x(z) = 1/z$.

Exercise 11. List all possible situations for L'Hospital's rule and write down detailed proof for each. (Hint: Some proofs can be obtained from others by replacing $f(x) \leftrightarrow -f(x)$ etc.)

Example 36. Find $\lim_{x \longrightarrow 0} x \ln x$.

We have

$$\lim_{x \longrightarrow 0} x \ln x = \lim_{x \longrightarrow 0} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0} \frac{1/x}{-1/x^2} = \lim_{x \longrightarrow 0} (-x) = 0. \tag{75}$$

Remark 37. L'Hospital's rule only applies to the situations $0/0$, $(\pm\infty)/(\pm\infty)$.

Exercise 12. Show that applying L'Hospital's rule to $\lim_{x \longrightarrow 0} \frac{\sin x}{1+x}$ leads to wrong result. Then explain which step in the proof breaks down for this limit.

Example 38. Let $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Prove that f is differentiable at 0 and find $f'(0)$.

Solution. We need to study the limit

$$\lim_{x \rightarrow 0} \frac{e^{-(1/x^2)} - 0}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-(1/x^2)}}{x}. \tag{76}$$

4. Taylor Expansion

4.1. Higher order derivative.

Definition 39. (Second and higher order derivatives) *Let $f(x)$ be differentiable on (a,b). Let $f'(x)$ be its derivative. If $f'(x)$ is differentiable at x_0, then we denote*

$$f''(x_0) := (f')'(x_0) \tag{77}$$

and say $f(x)$ is twice differentiable at x_0. We say $f(x)$ is twice differentiable on (a,b) if $f''(x)$ exists for all $x \in (a,b)$.

Similarly, for any $n > 2$, if $f^{(n-1)}(x)$ is differentiable at x_0, then we denote

$$f^{(n)}(x_0) := \left(f^{(n-1)}\right)'(x_0) \tag{78}$$

and say $f(x)$ is n-th differentiable at x_0. We say $f(x)$ is n-th differentiable on (a,b) if $f^{(n)}(x)$ exists for all $x \in (a,b)$.

Exercise 13. Let $n, m \in \mathbb{N}$. Let $f(x)$ be $n+m$-th differentiable at x_0. Prove that $f^{(n)}(x)$ is m-th differentiable at x_0 and furthermore

$$(f^{(n)})^{(m)}(x_0) = f^{(n+m)}(x_0). \tag{79}$$

(Hint: Induction.)

Example 40. Let $f(x) = e^{3x}$. Compute $f^{(n)}(x)$ for $n \in \mathbb{N}$.

We claim $f^{(n)}(x) = 3^n e^{3x}$ and prove this by induction. Denote by $P(n)$ the claim: $f^{(n)}(x) = 3^n e^{3x}$. Then

- $P(1)$: $f'(x) = 3\, e^{3x}$ thanks to chain rule. Therefore $P(1)$ holds.
- $P(n) \Longrightarrow P(n+1)$. Assume $P(n)$: $f^{(n)}(x) = 3^n e^{3x}$. Then by definition

$$f^{(n+1)}(x) = (3^n e^{3x})' = 3^n \cdot 3\, e^{3x} = 3^{(n+1)} e^{3x}. \tag{80}$$

Thus $P(n+1)$ holds and the proof ends.

Exercise 14. Let $f(x) = \sin x$. Calculate $f^{(n)}(x)$ for all $n \in \mathbb{N}$. Justify your answer.

Remark 41. Note that for $f^{(n)}(x_0)$ to exist, $f^{(n-1)}(x)$ must exist over $(x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$.

Exercise 15. Explain why existence of $f'(x_0)$ is not enough to define $f''(x_0)$.

Remark 42. (Notation) Usually we use f', f'', f''' to denote first, second, third order derivatives, while switch to $f^{(4)}$, $f^{(5)}$,... for higher order derivatives.

4.2. Taylor expansion.

Theorem 43. (Lagrange form of the remainder) *Let f be such that $f^{(k)}(x)$ exists on (a, b) for $k = 0, \ldots, n+1$. Then for every $x, x_0 \in (a, b)$ the following holds:*

$$f(x) = P_n(x) + R_n(x) \tag{81}$$

where

$$P_n(x) := f(x_0) + f'(x_0)\,(x - x_0) + \frac{f''(x_0)}{2}\,(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n. \tag{82}$$

is called the Taylor polynomial of f at x_0 to degree n, and

$$R_n(x) := f(x) - P_n(x) \tag{83}$$

is called the Remainder.

Under the assumptions of the theorem, we have the following formula for the remainder:

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1} \tag{84}$$

where ξ is between x, x_0. This formula is called the "Lagrange form of the remainder".

Remark 44. It is important to understand

- $P_n(x)$ depends on x_0, that is it changes when x_0 changes;
- For every fixed x_0, $P_n(x)$ is a polynomial. However the dependence of $P_n(x)$ on x_0 may be more complicated.
- The ξ in $R_n(x)$: $= \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}$ depends on both x and x_0, that is when x or x_0 changes, so does ξ. Furthermore the there is no convenient formula yielding ξ from x, x_0.
- One way to understand the role of ξ is as follows. For any fixed x, it is clear that there is $r \in \mathbb{R}$ such that

  $$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + r (x - x_0)^{n+1}. \tag{85}$$

  Thus what the theorem actually says is: $\exists \xi$ between x, x_0 such that $r = \frac{f^{(n+1)}(\xi)}{(n+1)!}$.

Proof. In the following x is fixed. Take $r \in \mathbb{R}$ such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2} (x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + r (x - x_0)^{n+1}. \tag{86}$$

holds for **this particular** x.

Now set

$$g(t) = f(t) - \left[f(x_0) + f'(x_0)(t - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} (t - x_0)^n + r (t - x_0)^{n+1} \right] \tag{87}$$

then we have $g(x_0) = g(x) = 0$. Applying Rolle's theorem, we obtain ξ_1 between x_0, x such that $g'(\xi_1) = 0$. On the other hand clearly $g'(x_0) = 0$. Thus we have ξ_2 between ξ_1 and x_0 (thus also between x, x_0) such that $g''(\xi_2) = 0$. Apply this n times we conclude that there is ξ such that $g^{(n+1)}(\xi) = 0$, which gives

$$r = \frac{f^{(n+1)}(\xi)}{(n+1)!}. \tag{88}$$

Thus ends the proof. □

Remark 45. Note that the case $n = 0$ is exactly Rolle's theorem. Also note that one cannot prove the above theorem through induction.

Remark 46. A more clever proof is as follows. Set

$$F(t) = f(x) - \left[f(t) + f'(t)(x - t) + \cdots + \frac{f^{(n)}(t)}{n!} (x - t)^n \right], \qquad G(t) = (x - t)^{n+1} \tag{89}$$

We have $F(x)=G(x)=0$, $F(x_0)=R_n(x)$, $G(x_0)=(x-x_0)^{n+1}$, and

$$F'(t)=-\frac{f^{(n+1)}(t)}{n!}(x-t)^n, \qquad G'(t)=-(n+1)(x-t)^n. \tag{90}$$

Now apply Cauchy's MVT: There is ξ between x_0, x and $\xi\neq x_0$, x (this can be written as $0<|\xi-x_0|<|x-x_0|$) such that

$$\frac{f^{(n+1)}(\xi)}{(n+1)!}=\frac{F'(\xi)}{G'(\xi)}=\frac{F(x)-F(x_0)}{G(x)-G(x_0)}=\frac{R_n(x)}{(x-x_0)^{n+1}}\Longrightarrow R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}. \tag{91}$$

Exercise 16. Try to prove the theorem through induction and explain the difficulty you encounter.

Exercise 17. Check out the youtube video (http://www.youtube.com/watch?v=LpWIy_eFp3M) "proving" Taylor expansion formula and explain where the mistake is. Compare it with the Khan Academy video on the same topic.

Exercise 18. Prove the theorem as follows. Fix x, x_0. Define

$$F(t)=f(t)-\left[f(x_0)+f'(x_0)(t-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(t-x_0)^n\right]; \qquad G(t)=(t-x_0)^{n+1}. \tag{92}$$

Apply Cauchy's MVT n times to $\frac{F(t)-F(x_0)}{G(t)-G(x_0)}$.

Example 47. Calculate Taylor expansion with Lagrange form of remainder (to degree 2 – that is $n=2$) of the following functions at $x_0=0$.

$$f(x)=\sin(\sin x); \qquad f(x)=x^4+x+1 \tag{93}$$

Solution.

- $f(x)=\sin(\sin x)$. We calculate:

$$f'(x)=[\cos(\sin x)]\cos x\Longrightarrow f'(0)=1; \tag{94}$$

$$f''(x)=[-\sin(\sin x)\cos x]\cos x-[\cos(\sin x)]\sin x\Longrightarrow f''(0)=0; \tag{95}$$

$$\begin{aligned} f'''(x) &= \{[-\sin(\sin x)]\cos^2 x\}'-\{[\cos(\sin x)]\sin x\}' \\ &= -\cos(\sin x)\cos^3 x+2\sin(\sin x)\cos x\sin x \\ &\quad +\sin(\sin x)\sin x\cos x-\cos(\sin x)\cos x \\ &= -\cos x\,[(\cos^2 x+1)\cos(\sin x)-3\sin x(\sin(\sin x))]. \end{aligned} \tag{96}$$

  Thus the Taylor polynomial at $x_0=0$ to degree 2 reads:

$$0+1\cdot(x-0)+\frac{0}{2}(x-0)^2+\frac{f'''(\xi)}{6}(x-0)^3 \tag{97}$$

  which simplifies to

$$\sin(\sin x)=x+\frac{-\cos\xi\,[(\cos^2\xi+1)\cos(\sin\xi)-3\sin\xi(\sin(\sin\xi))]}{6}x^3. \tag{98}$$

  Here ξ lies between 0 and x.

- $f(x)=x^4+x+1$. We calculate:

$$f(0)=1,\ f'(x)=4x^3+1\Longrightarrow f'(0)=1,\ f''(x)=12x^2\Longrightarrow f''(0)=0 \tag{99}$$

  and

$$f'''(x)=24x. \tag{100}$$

Therefore the Taylor polynomial at $x_0 = 0$ to degree 2 reads

$$x^4 + x + 1 = 1 + x + (4\,\xi)\,x^3 \tag{101}$$

where ξ lies between 0 and x.

Example 48. Calculate Taylor expansion (to degree 2) of the following functions at the specified x_0's.

$$f(x) = \sin x,\ x_0 = \frac{\pi}{2}; \qquad f(x) = x^4 + x + 1,\ x_0 = 1; \qquad f(x) = e^x,\ x_0 = 2. \tag{102}$$

Solution.

- $f(x) = \sin x, x_0 = \frac{\pi}{2}$.
  We have

$$f(x_0) = \sin\left(\frac{\pi}{2}\right) = 1; \tag{103}$$

$$f'(x) = \cos x \Longrightarrow f'(x_0) = 0; \tag{104}$$

$$f''(x) = -\sin x \Longrightarrow f''(x_0) = -1; \tag{105}$$

$$f'''(x) = -\cos x. \tag{106}$$

  Therefore the answer is

$$\sin x = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 - \frac{\cos\xi}{6}\left(x - \frac{\pi}{2}\right)^3 \tag{107}$$

  where ξ is between x and $\pi/2$.

- $f(x) = x^4 + x + 1,\ x_0 = 1$.
  We have

$$f(x_0) = 3; \qquad f'(x) = 4\,x^3 + 1 \Longrightarrow f'(x_0) = 5 \tag{108}$$

$$f''(x) = 12\,x^2 \Longrightarrow f''(x_0) = 12; \qquad f'''(x) = 24\,x. \tag{109}$$

  So the answer is

$$x^4 + x + 1 = 3 + 5\,(x-1) + 6\,(x-1)^2 + 4\,\xi\,(x-1)^3 \tag{110}$$

  where ξ is between x and 1.

- $f(x) = e^x,\ x_0 = 2$.
  We have

$$f(x_0) = f'(x_0) = f''(x_0) = e^2, \qquad f'''(x) = e^x. \tag{111}$$

  So the answer is

$$e^x = e^2 + e^2\,(x-2) + \frac{e^2}{2}\,(x-2)^2 + \frac{e^\xi}{6}\,(x-2)^3 \tag{112}$$

  where ξ is between x and 1.

Exercise 19. Find a computer system (the most convenient one would be www.wolframalpha.com) that can plot functions. Plot the Taylor polynomial to degrees $2, 3, 4, 5$ of $f(x) = \sin x$. Observe how they approximates $f(x)$ near x_0. What happens far away from x_0?

4.3. Applications of Taylor expansion.

Example 49. Prove the following.

a) $e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ for all $x > 0$.

b) $\left|\cos x - \left(1 - \frac{x^2}{2}\right)\right| < \frac{1}{24}$ for all $x \in (-1, 1)$.

Proof.

a) The Taylor polynomial with Lagrange remainder for e^x at 0 is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{e^\xi}{24} x^4. \tag{113}$$

Since $x > 0$, ξ (note that it depends on x, that is $\xi = \xi(x)$ is in fact a function of x) is also positive. Consequently $\frac{e^\xi}{24} x^4 > 0$ for all x. So $e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ holds for all $x > 0$.

b) The Taylor polynomial with Lagrange remainder for $\cos x$ at 0 is (up to degree 2):

$$\cos x = 1 - \frac{x^2}{2} + \frac{\sin \xi}{6} x^3 \tag{114}$$

with ξ between 0 and x. Thus we have

$$\left|\cos x - \left(1 - \frac{x^2}{2}\right)\right| = \frac{|\sin \xi|}{6} |x|^3 < \frac{1}{6}. \tag{115}$$

for all $x \in (-1, 1)$. This is not enough so we expand one more term:

$$\cos x = 1 - \frac{x^2}{2} + \frac{\cos \xi}{24} x^4 \tag{116}$$

which gives

$$\left|\cos x - \left(1 - \frac{x^2}{2}\right)\right| = \frac{|\cos \xi|}{24} |x|^4 < \frac{1}{24}. \tag{117}$$

Thus ends the proof. □