The Midterm Exam (Math 314 A1)

October 27, 2011

Name: ______________________ I.D.#: ______________

1. (16 points)

   (a) Let a, b, c, and d be positive real numbers. Prove that

$$\frac{a}{b} < \frac{c}{d} \quad \text{implies} \quad \frac{a+c}{b+d} < \frac{c}{d}.$$

   (b) Use the interval notation to express the set $\{x \in \mathbb{R} : |2x+1| \geq 5\}$.

2. (16 points)

(a) Let $E := \{(-1)^n + (2/3)^{n-1} : n \in \mathbb{N}\}$. Find $\max E$, $\sup E$, $\min E$, and $\inf E$. Justify your answer.

(b) Let A be a bounded nonempty subset of $\mathbb{R}$, and let $B := \{2x : x \in A\}$. Prove

$$\sup B = 2(\sup A).$$

3. (18 points) Find the following limits.

(a) $\lim_{n\to\infty} \left[\sqrt{n^2+2n} - (n+1)\right]$.

(b) $\lim_{x\to 2} \dfrac{x^2-5x+6}{x^2-4}$.

(c) For $n = 1, 2, \ldots$, let $s_n := a + ar + ar^2 + \cdots + ar^{n-1}$, where $-1 < r < 1$. Find $\lim_{n\to\infty} s_n$.

4. (18 points) Let $a_1 := 2$ and set

$$a_{n+1} := \frac{3a_n + 1}{5}, \quad n = 1, 2, \ldots.$$

(a) Use mathematical induction to prove that $a_n > 0$ for all $n \in \mathbb{N}$.

(b) Use mathematical induction to show that $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.

(c) As a bounded decreasing sequence, $(a_n)_{n=1,2,\ldots}$ converges. Find $\lim_{n\to\infty} a_n$.

5. (16 points)

(a) Let

$$b_n := \big[1 - (-1)^n\big] n + \frac{50}{n}, \quad n \in \mathbb{N}.$$

Find an increasing subsequence of $(b_n)_{n=1,2,\ldots}$. Also, find a convergent subsequence of $(b_n)_{n=1,2,\ldots}$.

(b) Let $(x_n)_{n=1,2,\ldots}$ be a sequence of real numbers. Suppose that

$$\big|x_{n+1} - x_n\big| \leq \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \quad \forall n \in \mathbb{N}.$$

Show that $(x_n)_{n=1,2,\ldots}$ is a Cauchy sequence.

6. (16 points) Let $f(x) := 1 - x^2$ for $x \in \mathbb{R}$, and let

$$g(x) := \begin{cases} 1 & \text{if } x \geq 1, \\ -1 & \text{if } x < 1. \end{cases}$$

(a) Prove that g is *not* continuous at 1.

(b) Prove that $f \circ g$ is continuous on $\mathbb{R}$, but $g \circ f$ is *not* continuous on $\mathbb{R}$.