The Final Exam (Math 314 A1)

December 20, 2011

Name: ______________________________ I.D.#: ____________________

1. (10 points) Let

$$f(x) := \frac{\sqrt{x+1}-1}{\sqrt{x}}, \quad x \in (0,1].$$

   (a) Find $\lim_{x \to 0^+} f(x)$.

   (b) Prove that there exists a point $c \in (0,1]$ such that $f(x) \leq f(c)$ for all $x \in (0,1]$.

2. (15 points) Let $g(x) := 2^x - 3x$, $-\infty < x < \infty$.

(a) Show that there exists some $a \in (0, 1)$ such that $g(a) = 0$.

(b) Find $g'(x)$ and show that there exists a unique $b \in (-\infty, \infty)$ such that $g'(b) = 0$. Give an explicit expression of b.

(c) Let b be as given in part (b). Show that $g(x) \geq g(b)$ for all $x \in (-\infty, \infty)$.

3. (10 points) (a) Let f be defined by

$$f(x) := \begin{cases} ax+b & \text{for } x < -1, \\ x^3+1 & \text{for } -1 \le x \le 2, \\ cx+d & \text{for } x > 2. \end{cases}$$

Determine the constants a, b, c, and d such that f is differentiable on $\mathbb{R}$.

(b) Let

$$g(x) := \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that g is differentiable at 0 and that $g'(0) = 0$.

4. (10 points) Let f be a differentiable function on an open interval (a, b), and let A and B be two real numbers such that $A < B$.

   (a) Suppose that $A \leq f'(x) \leq B$ for all $x \in (a, b)$. For $a < x_1 < x_2 < b$ prove that

$$A(x_2 - x_1) \leq f(x_2) - f(x_1) \leq B(x_2 - x_1).$$

   (b) Suppose that $A > 0$ and $f'(x) \geq A$ for all $x \in (a, b)$. Then f is a one-to-one function on (a, b) and its range is an open interval J. Let g denote the inverse function of f. Prove that

$$0 < g'(y) \leq \frac{1}{A} \quad \forall y \in J.$$

5. (10 points) Let $g(t) := \ln(1+t)$, $t > -1$.

(a) Find a cubic polynomial p such that $p^{(k)}(0) = g^{(k)}(0)$ for $k = 0, 1, 2, 3$. (Hint: p is the Taylor polynomial of third degree of g at 0.)

(b) Let $f(x) := g(x^2) = \ln(1+x^2)$, $-\infty < x < \infty$. Find a polynomial q of degree six such that $q^{(k)}(0) = f^{(k)}(0)$ for $k = 0, 1, 2, 3, 4, 5, 6$. Justify your answer.

6. (15 points) The Fundamental Theorem of Calculus will be used in this problem.

(a) Let $F(x) := \int_x^{x^2} \sqrt{1+t^2}\, dt$, $x \in \mathbb{R}$. Find $F'(x)$ for $x \in \mathbb{R}$.

(b) Let $G(x) := \int_0^x x \sin(t^2)\, dt$, $x \in \mathbb{R}$. Find $G''(x)$ for $x \in \mathbb{R}$.

(c) Let $H(x) := \int_x^{x+\pi/2} |\cos(2t)|\, dt$, $x \in \mathbb{R}$. Show that H is a constant and find this constant.

7. (15 points) Calculate the following integrals.

(a) $\displaystyle\int_0^2 x^2\sqrt{x^3+1}\,dx.$

(b) $\displaystyle\int_0^\pi \cos^2 x\,dx.$

(c) $\displaystyle\int_0^1 e^{\sqrt{x}}\,dx.$

8. (15 points) (a) It is known that $\dfrac{1-r^{n+1}}{1-r} = 1 + r + \cdots + r^n$ for $r \neq 1$ and all $n \in \mathbb{N}$. Derive from this identity the following formula for $x \in \mathbb{R}$:

$$\frac{1-(-x^2)^{n+1}}{1+x^2} = \sum_{k=0}^{n} (-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} \quad \forall n \in \mathbb{N}.$$

(b) For $n \in \mathbb{N}$, let $g_n(x) := \sum_{k=0}^{n} (-1)^k x^{2k}$. Use part (a) to prove

$$\left| g_n(x) - \frac{1}{1+x^2} \right| \leq x^{2n+2} \quad \text{for all } x \in [0,1].$$

(c) Use part (b) to prove

$$\lim_{n\to\infty} \int_0^1 g_n(x)\, dx = \int_0^1 \frac{1}{1+x^2}\, dx.$$

Moreover, find $\int_0^1 g_n(x)\, dx$ and the limit $\displaystyle\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{2k+1}$.