Math 314 Fall 2012 Midterm Practice Problems Solutions

Oct. 19, 2012

- To best prepare for midterm, also review homework problems.
- To get most out of these problems, *clearly write down* (instead of mumble or think) your *complete* answers (instead of a few lines of the main idea), in *full sentences* (instead of formulas connected by arrows). And then compare with the solutions when they are posted.
- If don't know where to start, write down all definitions involved.
- If have no idea what to do, try proof by contradiction. Start by writing down the assumption in logical statements.
- "Justify" means: if true, provide a proof; if false, give a counterexample.

Practice Problems

Problem 1. $f(x): E \mapsto \mathbb{R}$ is Hölder continuous if there are $a > 0$ and $M \in \mathbb{R}$ such that for every $x, y \in E$, $|f(x) - f(y)| \leqslant M\, |x - y|^a$. Write down the logical statement for "$f(x)$ is not Hölder continuous".
Solution. $f(x)$ is Hölder continuous if

$$\exists a > 0 \exists M \in \mathbb{R}\ \forall x, y \in E \qquad |f(x) - f(y)| \leqslant M\, |x - y|^a. \tag{1}$$

$f(x)$ is not Hölder continuous if

$$\forall a > 0\ \forall M \in \mathbb{R}\ \exists x, y \in E \qquad |f(x) - f(y)| > M\, |x - y|^a. \tag{2}$$

Note that x, y will change if a, M change.

Problem 2. Recall that $f(x)$ is increasing if $f(x_1) \geqslant f(x_2)$ whenever $x_1 \geqslant x_2$ Write down the logical statement for "$f(x)$ is not increasing".
Solution. $f(x)$ is increasing if

$$\forall x_1, x_2 \qquad x_1 \geqslant x_2 \Longrightarrow f(x_1) \geqslant f(x_2). \tag{3}$$

$f(x)$ is not increasing if

$$\exists x_1, x_2 \qquad \text{Not}\,[x_1 \geqslant x_2 \Longrightarrow f(x_1) \geqslant f(x_2)]. \tag{4}$$

This simplifies to (recall that $\text{Not}\,(A \Longrightarrow B) = \text{Not}\,[(\text{Not}\,A) \text{ or } B] = A \text{ and } (\text{Not}\,B)$)

$$\exists x_1, x_2,\ x_1 \geqslant x_2, \qquad f(x_1) < f(x_2). \tag{5}$$

Problem 3. Let A, B, C be logical statements. Prove that $[(A \Longrightarrow B) \text{ and } (B \Longrightarrow C)] \Longrightarrow (A \Longrightarrow C)$. Explain in English what this means.
Solution. We construct the truth table

A	B	C	$A \Longrightarrow B$	$B \Longrightarrow C$	$(A \Longrightarrow B)$ and $(B \Longrightarrow C)$	$A \Longrightarrow C$	$[(A \Longrightarrow B)$ and $(B \Longrightarrow C)] \Longrightarrow (A \Longrightarrow C)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Therefore the statement is always true. It means if A implies B and B implies C, then A implies C.

Problem 4. Let $f: X \mapsto Y$ be a function. Prove that f is one-to-one if and only if $f(A \backslash B) = f(A) \backslash f(B)$ for all subsets A, B of X.

Proof.

- "If". Assume $\forall A, B \subseteq X$, $f(A\backslash B) = f(A)\backslash f(B)$. For any $x_1 \neq x_2$, take $A = \{x_1, x_2\}$, $B = \{x_2\}$. Then $f(A\backslash B) = \{f(x_1)\}$, $f(A) = \{f(x_1), f(x_2)\}$, $f(B) = \{f(x_2)\}$. As $f(A)\backslash f(B) = \{f(x_1)\}$, $f(x_1) \neq f(x_2)$.
- "Only if". Assume f is one-to-one. We prove $f(A\backslash B) \subseteq f(A)\backslash f(B)$ and $f(A)\backslash f(B) \subseteq f(A\backslash B)$.
  - $f(A\backslash B) \subseteq f(A)\backslash f(B)$. Take any $y \in f(A\backslash B)$. By definition there is $x \in A\backslash B$ such that $y = f(x)$. $x \in A\backslash B$ means $x \in A, x \notin B$.

    Because $x \in A$, $y = f(x) \in f(A)$; On the other hand, since f is one-to-one and $x \notin B$, $y = f(x) \neq f(x')$ for any $x' \in B$ which means $y \notin f(B)$. Therefore $y \in f(A)\backslash f(B)$.
  - $f(A)\backslash f(B) \subseteq f(A\backslash B)$. Take any $y \in f(A)\backslash f(B)$. Then $y \in f(A), y \notin f(B)$. As $y \in f(A)$ there is $x \in A$ such that $y = f(x)$. Since $y \notin f(B)$, $x \notin B$. Therefore $x \in A\backslash B$ and consequently $y = f(x) \in f(A\backslash B)$. □

Problem 5. Suppose $f: A \mapsto B$ and $g: B \mapsto C$ are functions. Show that if both f and g are bijections, then so is $g \circ f$.

Proof. We need to show that

- $g \circ f$ is one-to-one. For any $x_1 \neq x_2$, since f is one-to-one, $f(x_1) \neq f(x_2)$. Now because g is one-to-one, $g(f(x_1)) \neq g(f(x_2))$.
- $g \circ f$ is onto. Take any $z \in C$. Since g is onto, there is $y \in B$ such that $z = g(y)$. Now because f is onto, there is $x \in A$ such that $y = f(x)$. Thus $z = g(f(x))$. □

Problem 6. Let

$$A = \left\{x \in \mathbb{R}: |\sin x| \leqslant \frac{1}{2}\right\}; \qquad B = \{x \in \mathbb{R}: x^3 - x^2 + x - 1 < 0\}. \tag{6}$$

a) Represent $A, B, A \cup B, A \cap B$ using intervals.

b) Which of these four sets is/are open? Which is/are closed? Justify your answers.

Solution.

a) $A = \cup_{n \in \mathbb{Z}}\left[n\,\pi - \frac{\pi}{6}, n\,\pi + \frac{\pi}{6}\right]$; $B = \{x \in \mathbb{R}: (x-1)\,(x^2+1) < 0\} = (-\infty, 1)$. $A \cup B = (-\infty, 1) \cup \left(\cup_{n \in \mathbb{N}}\left[n\,\pi - \frac{\pi}{6}, n\,\pi + \frac{\pi}{6}\right]\right)$; $A \cap B = \cup_{n=0}^{\infty}\left[-n\,\pi - \frac{\pi}{6}, -n\,\pi + \frac{\pi}{6}\right]$.

b)

- A is closed. Since $A^c = \cup_{n \in \mathbb{Z}}\left(n\,\pi + \frac{\pi}{6}, n\,\pi + \frac{5\,\pi}{6}\right)$ is open (because it is union of open intervals).
- B is open since it is an open interval.
- $C = A \cup B$ is neither open nor closed.
  - C is not open. Take $x_0 = \frac{5\,\pi}{6} \in C$. Then for any (a, b) such that $x_0 \in (a, b)$, there is $c > 0$ such that $\max\{1, a\} < c < x_0$. For this c we have $c \notin A \cup B$. Consequently $(a, b) \nsubseteq A \cup B$.
  - C is not closed. We have

$$(A \cup B)^c = \left[1, \frac{5\,\pi}{6}\right) \cup \left(\cup_{n=1}^{\infty}\left(n\,\pi + \frac{\pi}{6}, n\,\pi + \frac{5\,\pi}{6}\right)\right). \tag{7}$$

Now take $1\in(A\cup B)^c$. For any $(a,b)\ni 1$, we have $a<\frac{1+a}{2}<1$ and therefore $\frac{1+a}{2}\in(a,b)$ but $\frac{1+a}{2}\notin(A\cup B)^c$. Consequently $(a,b)\nsubseteq(A\cup B)^c$.

- $D=A\cap B$ is closed. Since $D^c=\left(\cup_{n=0}^{\infty}\left(-n\,\pi-\frac{5\,\pi}{6},-n\,\pi-\frac{\pi}{6}\right)\right)\cup\left(\frac{\pi}{6},\infty\right)$ is union of open intervals and is therefore open.

Problem 7. Decide which of the following statements are true and which are false. Prove the true ones and provide a counterexample for the false ones.

a) If $\{x_n\}$ is Cauchy and $\{y_n\}$ is bounded, then $\{x_n\,y_n\}$ is Cauchy;

b) If $\{x_n\}$ is a sequence of real numbers that satisfies $x_{2^k}-x_{2^{k-1}}\longrightarrow 0$ as $k\longrightarrow\infty$ and $x_n=0$ for all $n\neq 2^k$, $k\in\mathbb{N}$, then $\{x_n\}$ is Cauchy;

c) If $\{x_n\}$ and $\{y_n\}$ are Cauchy and $y_n\neq 0$ for all $n\in\mathbb{N}$, then $\{x_n/y_n\}$ is Cauchy;

d) If $\{x_n\}$ and $\{y_n\}$ are Cauchy, then $\{1/(|x_n|+|y_n|)\}$ cannot converge to zero.

Solution.

a) False. Take $x_n=1$ and $y_n=(-1)^n$ for all $n\in\mathbb{N}$. Now take $\varepsilon_0=1$. For any $N\in\mathbb{N}$, take $n>N$. Then

$$|x_n\,y_n-x_{n+1}\,y_{n+1}|=2>\varepsilon_0. \tag{8}$$

b) False. Take $x_{2^k}=1$ for all $k\in\mathbb{N}$. Clearly $x_{2^k}-x_{2^{k-1}}\longrightarrow 0$. Now take $\varepsilon_0=1$. For any $N\in\mathbb{N}$, take $n=2^k>N$. Then

$$|x_n-x_{n+1}|=1\geqslant\varepsilon_0. \tag{9}$$

c) False. Take $x_n=1$ and $y_n=1/n$ for all $n\in\mathbb{N}$. We have $\{x_n/y_n\}=n$ is not Cauchy.

d) True. Assume the contrary, that is $1/(|x_n|+|y_n|)\longrightarrow 0$ as $n\longrightarrow\infty$. As $\{x_n\},\{y_n\}$ are Cauchy, they are bounded. That is there are M_1,M_2 such that

$$|x_n|\leqslant M_1,\qquad |y_n|\leqslant M_2 \tag{10}$$

for all $n\in\mathbb{N}$.

Now due to $1/(|x_n|+|y_n|)\longrightarrow 0$, there is $N\in\mathbb{N}$ such that for all $n>N$, $1/(|x_n|+|y_n|)<1/(M_1+M_2+1)\Longrightarrow|x_n|+|y_n|>M_1+M_2+1$. Contradiction.

Problem 8. Prove that $x_n=\sum_{n=1}^{\infty}\frac{1}{n^2}$ is Cauchy.

Proof. For any $\varepsilon>0$, take $N\in\mathbb{N}$ such that $N>1/\varepsilon$. Then for any $n,m>N$, we have (without loss of generality take $m>n$)

$$|x_n-x_m|=\left|\frac{1}{(n+1)^2}+\cdots+\frac{1}{m^2}\right|\leqslant\left|\frac{1}{n\,(n+1)}+\cdots+\frac{1}{(m-1)\,m}\right|=\left|\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+1}-\frac{1}{n+2}+\cdots+\frac{1}{m-1}-\frac{1}{m}\right|\leqslant\frac{1}{n}<\frac{1}{N}<\varepsilon. \tag{11}$$

□

Problem 9. Prove that $x_n=\frac{(-1)^n\,(n-1)}{n+1}$ does not converge.

Proof. We have $x_{2n}=\frac{n-1}{n+1}=1-\frac{2}{n+1}\longrightarrow 1$ and $x_{2n+1}=-\frac{n-1}{n+1}=-1+\frac{2}{n+1}\longrightarrow -1$. Thus there are two subsequences converging to 1 and -1 respectively. Consequently $\{x_n\}$ does not converge. □

Problem 10. Let $x_n=\sum_{k=1}^{n}\frac{1}{\sqrt{k+1}+\sqrt{k}}$ for $n\in\mathbb{N}$. Does $\{x_n\}$ converge? Justify your answer.

Proof. We have

$$\frac{1}{\sqrt{k+1}+\sqrt{k}}=\frac{\sqrt{k+1}-\sqrt{k}}{\left(\sqrt{k+1}+\sqrt{k}\right)\left(\sqrt{k+1}-\sqrt{k}\right)}=\sqrt{k+1}-\sqrt{k}. \tag{12}$$

Therefore

$$x_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}} = \sum_{k=1}^{n} \left(\sqrt{k+1}-\sqrt{k}\right) = \sqrt{n+1}-1. \tag{13}$$

Take $\varepsilon_0 = 1$. For any $N \in \mathbb{N}$, take $n = N+1$ and $m = 4\,N+7$. We have

$$|x_m - x_n| = |\sqrt{m+1}-\sqrt{n+1}| = \left|\sqrt{4\,(N+2)}-\sqrt{N+2}\right| = \sqrt{N+2} > \varepsilon_0. \tag{14}$$

Thus $\{x_n\}$ is not Cauchy and therefore cannot converge. □

Problem 11. Let w_n satisfy $|w_n| \leqslant n^2$. Determine whether the following sequences are converging or not. If converging find the limit.

$$x_n = \sqrt{n^2+3\,n} - n - 3; \qquad y_n = \frac{e^n - 1}{3^n - 2^n}; \qquad z_n = \frac{w_n^2 + 4\,w_n + 5}{n^4 + 3\,n}. \tag{15}$$

You can use the fact that $r^n \longrightarrow 0$ as $n \longrightarrow \infty$ if $|r| < 1$.
Solution.

- $\{x_n\}$. We have

$$x_n = \frac{\left[\sqrt{n^2+3\,n}-(n+3)\right]\left[\sqrt{n^2+3\,n}+(n+3)\right]}{\left[\sqrt{n^2+3\,n}+(n+3)\right]} = \frac{-3\,n-9}{\sqrt{n^2+3\,n}+(n+3)} = \frac{-3-9/n}{\sqrt{1+3/n}+(1+3/n)}. \tag{16}$$

  Since $9/n \longrightarrow 0, 3/n \longrightarrow 0$, we have $-3-9/n \longrightarrow -3$, $\sqrt{1+3/n}+(1+3/n) \longrightarrow 2$. Because $2 \neq 0$, we have $x_n \longrightarrow -3/2$.

- $\{y_n\}$. We have

$$y_n = \frac{e^n-1}{3^n-2^n} = \frac{(e/3)^n - (1/3)^n}{1-(2/3)^n}. \tag{17}$$

  Since $|e/3|, |1/3|, |2/3| < 1$, we have $(e/3)^n, (1/3)^n, (2/3)^n \longrightarrow 0$ and consequently $y_n \longrightarrow \frac{0-0}{1-0} = 1$.

- $\{z_n\}$. We have

$$z_n = \frac{(w_n/n^2)^2 + 4\,(w_n/n^4) + 5/n^4}{1+3/n^3}. \tag{18}$$

  As $|w_n| \leqslant n^2$, we have $|w_n/n^4| \leqslant 1/n^2 \longrightarrow 0$ due to Squeeze theorem. However $|w_n/n^2| \leqslant 1$ is only bounded.

  In fact $\{z_n\}$ does not necessarily converge: Take $w_n = \left[\frac{1+(-1)^n}{2}\right] n^2$, we have $z_{2n} \longrightarrow 4$ and $z_{2n+1} \longrightarrow 0$.

Problem 12. Let $x_0 = 7$ and define x_n iteratively through

$$x_{n+1} = \frac{2\,x_n}{3} - 1. \tag{19}$$

Does $\lim_{n\to\infty} x_n$ exist? If it does, find the limit. Justify your answers.
Solution. Yes. We show that the sequence is Cauchy.

Subtracting $x_n = \frac{2\,x_{n-1}}{3} - 1$ from $x_{n+1} = \frac{2\,x_n}{3} - 1$ we have

$$x_{n+1} - x_n = \frac{2}{3}\,(x_n - x_{n-1}) \Longrightarrow |x_{n+1} - x_n| = \frac{2}{3}\,|x_n - x_{n-1}| \tag{20}$$

This leads to

$$|x_{n+1} - x_n| = \frac{2}{3}\,|x_n - x_{n-1}| = \left(\frac{2}{3}\right)^2 |x_{n-1} - x_{n-2}| = \cdots = \left(\frac{2}{3}\right)^n |x_1 - x_0|. \tag{21}$$

Consequently for any $\varepsilon > 0$, we can take $N \geqslant \log_{3/2}\left[\frac{3\,|x_1 - x_0|}{\varepsilon}\right]$, then for $m > n > N$,

$$\begin{aligned} |x_m - x_n| &\leqslant |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n| \\ &\leqslant \left[\left(\frac{2}{3}\right)^{m-1} + \cdots + \left(\frac{2}{3}\right)^n\right] |x_1 - x_0| \\ &= \left(\frac{2}{3}\right)^n \left[1 + \cdots + \left(\frac{2}{3}\right)^{m-n-1}\right] |x_1 - x_0| \\ &= \left(\frac{2}{3}\right)^n \frac{1 - \left(\frac{2}{3}\right)^{m-n}}{1 - \frac{2}{3}} |x_1 - x_0| \\ &\leqslant \left(\frac{2}{3}\right)^n 3\,|x_1 - x_0| \\ &< \left(\frac{2}{3}\right)^N 3\,|x_1 - x_0| < \varepsilon. \end{aligned} \tag{22}$$

Thus $\{x_n\}$ is Cauchy.

To find the limit, take $n \longrightarrow \infty$ on both sides we reach

$$\lim_{n \longrightarrow \infty} x_n = \frac{2}{3} \lim_{n \longrightarrow \infty} x_n - 1 \Longrightarrow \lim_{n \longrightarrow \infty} x_n = -3. \tag{23}$$

Problem 13. Let $\{x_n\}$ be a sequence of real numbers. Prove:

a) $\{x_n\}$ is unbounded if and only if there is a subsequence $\{x_{n_k}\}$ satisfying $|x_{n_k}| \to \infty$.

b) $\{x_n\}$ is bounded if and only if $\exists M \in \mathbb{R}$ such that $\limsup_{n \longrightarrow \infty} x_n \leqslant M$ and $\liminf_{n \to \infty} x_n \geqslant -M$.

Proof.

a) Since $\{x_n\}$ is unbounded, there is $n_1 \in \mathbb{N}$ such that $|x_{n_1}| \geqslant 1$. Now take $n_2 \in \mathbb{N}$ such that $|x_{n_2}| \geqslant |x_{n_1}| + 2$. In general, take x_{n_k} such that

$$|x_{n_k}| \geqslant |x_{n_{k-1}}| + k. \tag{24}$$

Thus we obtain a subsequence $\{x_{n_k}\}$ such that $|x_{n_k}| \geqslant k$. Now for any $M \in \mathbb{R}$, take $N \in \mathbb{N}$ such that $N > |M|$. Then for all $k > N$, $|x_{n_k}| \geqslant k > N > |M|$. Therefore $|x_{n_k}| \longrightarrow \infty$.

For the "if" part, we need to show for any $M \in \mathbb{R}$ there is $n \in \mathbb{N}$ such that $|x_n| > M$. But this follows directly from the existence of $\{x_{n_k}\}$ satisfying $|x_{n_k}| \longrightarrow \infty$, which by definition says: for any $M \in \mathbb{R}$ there is $K \in \mathbb{N}$ such that for all $k > K$, $|x_{n_k}| > M$.

b) Since $\{x_n\}$ is bounded, there is $M \in \mathbb{R}$ such that $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. This leads to

$$-M \leqslant \inf_{k \geqslant n} x_k \leqslant \sup_{k \geqslant n} x_k \leqslant M \tag{25}$$

for all $n \in \mathbb{N}$. Taking $n \longrightarrow \infty$ we conclude $\limsup_{n \longrightarrow \infty} x_n \leqslant M$ and $\liminf_{n \to \infty} x_n \geqslant -M$ thanks to comparison theorem.

For the "if" part, we need to show there is $M' \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $|x_n| \leqslant M'$. By definition, $\limsup_{n \longrightarrow \infty} x_n \leqslant M$ means there is $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $\sup_{k \geqslant n} x_n \leqslant M + 1$. In particular we have $\sup\{x_{N_1+1}, x_{N_1+2}, \ldots\} \leqslant M + 1$ which leads to

$$\forall n > N_1, x_n \leqslant M + 1. \tag{26}$$

Apply similar argument to $\liminf_{n \longrightarrow \infty} x_n \geqslant -M$, we obtain $N_2 \in \mathbb{N}$ such that

$$\forall n > N_2, \qquad x_n \geqslant -M - 1. \tag{27}$$

Now set $N = \max\{N_1, N_2\}$. We have (note that $M \geqslant \limsup x_n \geqslant \liminf x_n \geqslant -M \Longrightarrow M \geqslant 0$)

$$\forall n > N, \qquad |x_n| \leqslant M + 1 \tag{28}$$

Finally take

$$M' = \max\{|x_1|, |x_2|, \ldots, |x_N|, M + 1\}. \tag{29}$$

We have $|x_n| \leqslant M'$ for all $n \in \mathbb{N}$. □

Problem 14. Let A be a nonempty subset of $\mathbb{R}$. Let $B = 3\,A := \{3\,x : x \in A\}$. Derive the relations between $\sup B$, $\inf B$ and $\sup A$, $\inf A$. Justify your answers. Note that you may need to discuss different cases for c and for $\sup A$.

Solution. We prove $\sup B = 3 \sup A$. We only need to show:

1. $3 \sup A$ is an upper bound of B. For any $b \in B$, by definition there is $a \in A$ such that $b = 3\,a$. By definition of sup we have $\sup A \geqslant a \Longrightarrow 3 \sup B \geqslant 3\,a = b$.
2. $3 \sup A$ is the smallest upper bound of B. Let c be an upper bound of B. There is $c \geqslant b$ for all $b \in B$. By definition of B we have $c \geqslant 3\,a$ for all $a \in A$. Consequently $c/3 \geqslant a$ for all $a \in A$. By definition of sup we have $c/3 \geqslant \sup A \Longrightarrow c \geqslant 3 \sup A$.

$\inf B = 3 \inf A$ can be proved similarly.

Problem 15. Let $x_n = (-1)^{3n} + \frac{1}{n^2}$ for $n \in \mathbb{N} = \{1, 2, \ldots\}$. Let $A := \{x_n : n \in \mathbb{N}\} = \{x_1, x_2, \ldots\}$.

a) Find $\max A, \sup A, \min A, \inf A$. Justify your answers.

b) Find $\limsup_{n \to \infty} x_n$ and $\liminf_{n \to \infty} x_n$. Justify your answers.

Solution.

a)

- $\max A = 5/4$. To justify, we show
  1. $5/4 \in A$. This is clear since $5/4 = x_2$.
  2. $5/4 \geqslant a$ for every $a \in A$. Take any $x_n \in A$. If $n = 1$, we have $x_1 = -1 + 1 = 0 < 5/4$. If $n \geqslant 2$ we have $x_n = (-1)^{3n} + 1/n^2 \leqslant 1 + 1/4 = 5/4$.
- Since $\max A$ exists, $\sup A = \max A = 5/4$.
- $\min A$ does not exist. Assume the contrary. Let $x_{n_0} = \min A$. But then we have

$$x_{n_0+2} = (-1)^{3(n_0+2)} + \frac{1}{(n_0+2)^2} < (-1)^{3n_0} + \frac{1}{n_0^2} = x_{n_0} \tag{30}$$

  Contradiction.
- $\inf A = -1$. We show
  1. -1 is a lower bound. We have

$$x_n = (-1)^{3n} + \frac{1}{n^2} \geqslant (-1)^{3n} \geqslant -1 \tag{31}$$

     for all $n \in \mathbb{N}$.
  2. -1 is the largest lower bound. Let b be a lower bound that is $b \leqslant (-1)^{3n} + \frac{1}{n^2}$. Thus in particular, we have

$$b \leqslant (-1)^{3(2k+1)} + \frac{1}{(2\,k+1)^2} = -1 + \frac{1}{(2\,k+1)^2} \tag{32}$$

     for all $k \in \mathbb{N}$. Taking $k \longrightarrow \infty$, by comparison theorem we have $b \leqslant -1$.

b) For any $n \in \mathbb{N}$, we have

$$\sup_{k \geqslant n} x_k = \sup \left\{ (-1)^{3k} + \frac{1}{k^2} : k \geqslant n \right\} \leqslant 1 + \frac{1}{n^2} \tag{33}$$

because when $k \geqslant n$,

$$(-1)^{3k} \leqslant 1, \qquad \frac{1}{k^2} \leqslant \frac{1}{n^2}; \tag{34}$$

On the other hand, taking $k = 2\,n$ we have

$$(-1)^{3k} + \frac{1}{k^2} \geqslant 1 \Longrightarrow \sup_{k \geqslant n} x_k \geqslant 1. \tag{35}$$

Thus by comparison theorem

$$1 \leqslant \limsup_{n \longrightarrow \infty} x_n = \lim_{n \longrightarrow \infty} \left[\sup_{k \geqslant n} x_k \right] \leqslant \lim_{n \longrightarrow \infty} \left[1 + \frac{1}{n^2} \right] = 1. \tag{36}$$

Therefore

$$\limsup_{n \longrightarrow \infty} x_n = 1. \tag{37}$$

Similarly, for any $n \in \mathbb{N}$, we have

$$-1 \leqslant \inf_{k \geqslant n} x_k \leqslant -1 + \frac{1}{n^2}. \tag{38}$$

Taking limit $n \longrightarrow \infty$ and apply comparison theorem, we conclude

$$\liminf_{n \longrightarrow \infty} x_n = -1/ \tag{39}$$

Problem 16. Let $f(x)$ be a continuous function and let $a \in \mathbb{R}$. Prove that the pre-image $f^{-1}(a)$ is a closed set.

Proof. If $f^{-1}(c) = \varnothing$ then by definition it is open.

All we need to show is $[f^{-1}(a)]^c$ is open. By definition

$$[f^{-1}(a)]^c = \{x \in \mathbb{R}: f(x) = a\}^c = \{x \in \mathbb{R}: f(x) \neq a\}. \tag{40}$$

Take any $x_0 \in \{x \in \mathbb{R}: f(x) \neq a\}$. Set $\varepsilon = |f(x_0) - a|$. Then by the continuity of f there is $\delta > 0$ such that for all $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon = |f(x_0) - a|$. This implies for all such x, $f(x) \neq a$, or equivalently $(x_0 - \delta, x_0 + \delta) \subseteq \{x \in \mathbb{R}: f(x) \neq a\}$. Therefore $[f^{-1}(a)]^c$ is open. □

Problem 17. Find the limits of

$$\lim_{x \longrightarrow 3} \frac{x^2 - 5x + 6}{x^2 - 9}; \qquad \lim_{x \longrightarrow 0} \frac{\sqrt{1 + x} - 1}{x}; \qquad \lim_{x \longrightarrow \infty} \frac{x^3 + 5x + 6}{\sqrt{x^6 + 3x} - 7}. \tag{41}$$

Indicate clearly what property you are using at each step.

Solution.

- First we simplify

$$\frac{x^2 - 5x + 6}{x^2 - 9} = \frac{(x - 2)(x - 3)}{(x - 3)(x + 3)} = \frac{x - 2}{x + 3}. \tag{42}$$

Since polynomials are continuous everywhere, we have

$$x - 2 \longrightarrow 3 - 2 = 1, \qquad x + 3 \longrightarrow 3 + 3 = 6. \tag{43}$$

Since $6 \neq 0$ the limit exists and equals $1/6$.

- First simplify

$$\frac{\sqrt{1 + x} - 1}{x} = \frac{(\sqrt{1 + x} - 1)(\sqrt{1 + x} + 1)}{x(\sqrt{1 + x} + 1)} = \frac{x}{x(\sqrt{1 + x} + 1)} = \frac{1}{\sqrt{1 + x} + 1}. \tag{44}$$

We know that $\sqrt{x}$ is a continuous function for $x \geqslant 0$, therefore

$$\lim_{x \longrightarrow 0} \frac{1}{\sqrt{1 + x} + 1} = \frac{\lim_{x \longrightarrow 0} 1}{\lim_{x \longrightarrow 0} (\sqrt{1 + x} + 1)} = \frac{1}{2}. \tag{45}$$

The last step is because $\lim \sqrt{1 + x} + 1 = 2 \neq 0$.

- We have

$$\frac{x^3 + 5x + 6}{\sqrt{x^6 + 3x} - 7} = \frac{1 + 5x^{-2} + 6x^{-3}}{\sqrt{1 + 3x^{-5}} - 7x^{-3}} \longrightarrow \frac{1 + 0 + 0}{1 - 0} = 1. \tag{46}$$

Problem 18. Find and prove the limit

$$\lim_{x \longrightarrow -\infty} \left(\sqrt{x^2 + 2x} + x \right). \tag{47}$$

Proof. We have

$$\sqrt{x^2+2x}+x=\frac{\left(\sqrt{x^2+2x}+x\right)\left(\sqrt{x^2+2x}-x\right)}{\sqrt{x^2+2x}-x}=\frac{2x}{\sqrt{x^2+2x}-x}=\frac{2}{-\sqrt{1+2/x}-1}\longrightarrow -1. \tag{48}$$

□

Problem 19. Let $f(x)\colon (0,2)\mapsto \mathbb{R}$ be defined as

$$f(x)=\begin{cases}\dfrac{x^2-1}{x^2-3x+2} & x\neq 1\\ c & x=1\end{cases}. \tag{49}$$

Find all $c\in\mathbb{R}$ which makes $f(x)$ continuous at $x=1$. Justify your answer.

Proof. First we simplify

$$\frac{x^2-1}{x^2-3x+2}=\frac{(x-1)(x+1)}{(x-2)(x-1)}=\frac{x+1}{x-2}. \tag{50}$$

Therefore

$$\lim_{x\longrightarrow 1} f(x)=-2. \tag{51}$$

Consequently $c=-2$ is the only value that makes $f(x)$ continuous at $x=0$. □

Problem 20. Let $f\colon \mathbb{R}\mapsto\mathbb{R}$ be a real function. Prove that $f(x)\longrightarrow 0$ if and only if $|f(x)|\longrightarrow 0$. Is it true that $f(x)\longrightarrow L\neq 0$ if and only if $|f(x)|\longrightarrow |L|$? Justify your answer.

Proof.

- $f(x)\longrightarrow 0 \Longrightarrow |f(x)|\longrightarrow 0$.
  For any $\varepsilon>0$, since $f(x)\longrightarrow 0$ there is $\delta>0$ such that for all $|x-x_0|<\delta$, $|f(x)-0|<\varepsilon$. But this is just the definition of $|f(x)|\longrightarrow 0$ as $x\longrightarrow x_0$.
- $|f(x)|\longrightarrow 0\Longrightarrow f(x)\longrightarrow 0$.
  For any $\varepsilon>0$, since $|f(x)|\longrightarrow 0$ there is $\delta>0$ such that for all $|x-x_0|<\delta$, $||f(x)|-0|<\varepsilon$. Thus $|f(x)-0|=||f(x)|-0|<\varepsilon$.
- $f(x)\longrightarrow L\neq 0\Longrightarrow |f(x)|\longrightarrow |L|$ is true.
  For any $\varepsilon>0$, since $f(x)\longrightarrow L$, there is $\delta>0$ such that for all $|x-x_0|<\delta$, $|f(x)-L|<\varepsilon$. Since $||f(x)|-|L||\leqslant |f(x)-L|$, we conclude that $|f(x)|\longrightarrow |L|$.
- $|f(x)|\longrightarrow |L|\Longrightarrow f(x)\longrightarrow L$ is false. A counterexample is $f(x)=\begin{cases}1 & x>0\\ -1 & x\leqslant 0\end{cases}$. Then $|f(x)|=1$ a constant function, therefore $|f(x)|\longrightarrow 1$ as $x\longrightarrow 0$. But $\lim_{x\longrightarrow 0} f(x)$ does not exist, because if we take $\{x_n\}=\frac{(-1)^n}{n}$, then $x_n\longrightarrow 0$ but $f(x_n)=(-1)^n$ does not converge. □

HARDER PROBLEMS

- Problems at this level may or may not appear in the midterm.
- The solutions for these problems are sketchy. You are discouraged to read the solution before having seriously worked on the problems.

Problem 21. Let $A_n:=\left(1,1+\frac{3}{n^2}\right)$, $B_n:=\left[1,1+\frac{3}{n^2}\right]$, $C_n:=\left[1,1+\frac{3}{n^2}\right)$, $D_n:=\left(1,1+\frac{3}{n^2}\right]$ for every $n\in\mathbb{N}$.

a) Represent $\cup_{n=1}^{\infty}A_n$, $\cap_{n=1}^{\infty}A_n$, $\cup_{n=1}^{\infty}B_n$, $\cap_{n=1}^{\infty}B_n$ $\cup_{n=1}^{\infty}C_n$, $\cap_{n=1}^{\infty}C_n$ $\cup_{n=1}^{\infty}D_n$, $\cap_{n=1}^{\infty}D_n$ using intervals.

b) Among these eight sets, which is/are open? Which is/are closed? Justify your answers.

Solution. $(1,4),\varnothing,[1,4],\{1\}=[1,1],[1,4),\{1\}=[1,1],(1,4],\varnothing$.
$\cup A_n$, $\cap A_n, \cap D_n$ are open, $\cap A_n, \cup B_n, \cap B_n, \cap C_n, \cap D_n$ are closed.

Problem 22. Let $\{x_n\}$ be a sequence such that the subsequences $\{x_{2n}\}$, $\{x_{2n+1}\}$, $\{x_{3n}\}$ are convergent. Show that $\{x_n\}$ is convergent.

Proof. Let $x_{2n} \longrightarrow a, x_{2n+1} \longrightarrow b$. First we show that $a = b$. Assume otherwise. Then we have $x_{3(2k)} \longrightarrow a$, $x_{3(2k+1)} \longrightarrow b$, but this contradicts the condition that x_{3n} is convergent. Therefore $a = b$.

Now we prove $x_n \longrightarrow a$. For any $\varepsilon > 0$, there are $N_1, N_2 \in \mathbb{N}$ such that for all $n > N_1$, $|x_{2n} - a| < \varepsilon$; For all $n > N_2$, $|x_{2n+1} - a| < \varepsilon$. Now take $N = \max\{2\,N_1, 2\,N_2 + 1\}$. Then for every $n > N$. we have $|x_n - a| < \varepsilon$. □

Problem 23. Let $f(x)$ satisfy: $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x_1,\ x_2$ satisfying $x_1,\ x_2 \in (x_0 - \delta,\ x_0 + \delta) \backslash \{x_0\}$, $|f(x_1) - f(x_2)| < \varepsilon$. Prove that $\lim_{x \rightarrow x_0} f(x)$ exists.

Proof. Take any $x_n \longrightarrow x_0$ with $x_n \neq x_0$. Then there is $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in (x_0 - \delta, x_0 + \delta) \backslash \{x_0\}$. Thus we see that for every $x_n \longrightarrow x_0$, $f(x_n)$ is Cauchy. All we need to show is that $\lim f(x_n)$ is the same for all $\{x_n\}$.

Assume the contrary. Then there is $x_n \longrightarrow x_0, y_n \longrightarrow x_0$ such that $f(x_n) \longrightarrow a$, $f(y_n) \longrightarrow b \neq a$. Now consider the new sequence $\{f(x_n)\} \cup \{f(y_n)\}$. This new sequence has two subsequences converging to different limits and thus cannot be Cauchy. Contradiction. □

Problem 24. Let $E \subseteq \mathbb{R}$ be nonempty. For every $x \in \mathbb{R}$, its distance to E is defined as

$$d(x) := \inf_{y \in E} |x - y|. \tag{52}$$

a) Prove that $d(x)$ is a continuous function.

b) Prove that inf can be replaced by min if and only if E is closed.

Proof.

a) Fix any $x_0 \in \mathbb{R}$. Take $x_n \in E$ such that

$$|x - x_n| \longrightarrow \inf_{y \in E} |x - y|, \qquad n \longrightarrow \infty. \tag{53}$$

For any $\varepsilon > 0$, take $\delta = \varepsilon$. Then we have, for any $|x - x_0| < \delta$,

$$d(x) - |x_0 - x_n| \leqslant |x - x_n| - |x_0 - x_n| \leqslant |x - x_0| < \delta. \tag{54}$$

Taking limit $n \longrightarrow \infty$, we reach

$$d(x) - d(x_0) < \delta. \tag{55}$$

Now repeat the above argument with x, x_0 switched, we obtain

$$d(x_0) - d(x) < \delta. \tag{56}$$

Therefore

$$|d(x) - d(x_0)| < \delta = \varepsilon. \tag{57}$$

b)

- "If". Assume E is closed. For any $x \in \mathbb{R}$, take $x_n \in E$ such that $|x - x_n| \longrightarrow d(x)$. As $|x - x_n|$ is convergent, it is bounded, that is there is $M \in \mathbb{R}$ such that $|x - x_n| \leqslant M$ for all $n \in \mathbb{N}$. Consequently $|x_n| \leqslant |x| + M$ for all $n \in \mathbb{N}$. By Bolzano-Weierstrass, there is a subsequence $x_{n_k} \longrightarrow \xi \in \mathbb{R}$. As E is closed, $\xi \in E$. Then we have $d(x) = \lim |x - x_{n_k}| = |x - \xi|$.
- "Only if". Assume the contrary, that is E is not closed. Then E^c is not open. Consequently there is $x_0 \in E^c$ such that for every $n \in \mathbb{N}$, $\left(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right) \not\subseteq E^c$, in other words there is $x_n \in E$ such that $|x_n - x_0| < \frac{1}{n}$. Consequently $d(x_0) = 0$. Now since $\min_{y \in E} |x_0 - y|$ exists, ther is $y_0 \in E$ such that $|x_0 - y_0| = 0 \Longrightarrow x_0 = y_0 \in E$. Contradiction. □

Problem 25. Let $f(x) \colon \mathbb{R} \mapsto \mathbb{R}$ be a real function. Define $f^+(x) = \lim_{n \longrightarrow \infty} \sup_{|x-y| < 1/n} f(y)$, $f^-(x) = \lim_{n \longrightarrow \infty} \inf_{|x-y| < 1/n} f(y)$. Prove that f is continuous at x_0 if and only if $f^+(x_0) = f^-(x_0) = f(x_0)$.

Proof.

- "If". Assume f is continuous. Then for every $\varepsilon > 0$, there is $\delta > 0$ such that for all $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon$. Thus for all $n > 1/\delta$, we have
$$f(x_0) - \varepsilon \leqslant \inf_{|x-y|<1/n} f(y) \leqslant \sup_{|x-y|<1/n} f(y) \leqslant f(x_0) + \varepsilon. \quad (58)$$
Taking limit and apply comparison theorem, we conclude
$$f(x_0) - \varepsilon \leqslant f^-(x_0) \leqslant f^+(x_0) \leqslant f(x_0) + \varepsilon. \quad (59)$$
As this is true for all $\varepsilon > 0$, we conclude $f^+(x_0) = f^-(x_0) = f(x_0)$.
- "Only if". For any $\varepsilon > 0$, since $f^+(x_0) = f(x_0)$, there is $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $\sup_{|x-x_0|<1/n} f(x) < f(x_0) + \varepsilon$, that is $f(x) < f(x_0) + \varepsilon$ for all $|x - x_0| < 1/n$. Similarly, there is $N_2 \in \mathbb{N}$ such that for all $n > N_2$, $f(x) > f(x_0) - \varepsilon$ for all $|x - x_0| < 1/n$.

  Now take $\delta = \min\left\{\frac{1}{N_1+1}, \frac{1}{N_2+1}\right\}$. Then for all $|x - x_0| < \delta$, we have $f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$ which means $|f(x) - f(x_0)| < \varepsilon$. □

Problem 26. Let $f(x): \mathbb{R} \mapsto \mathbb{R}$ be a real function. Prove that f is continuous if and only if for every open set $A \subseteq \mathbb{R}$, the pre-image $f^{-1}(A)$ is open.

Proof.

- "If". For any $\varepsilon > 0$, the set $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ is open. Therefore its preimage $\{x \in \mathbb{R}: |f(x) - f(x_0)| < \varepsilon\}$ is open. Consequently there is (a, b) such that $x_0 \in (a, b) \subset \{x \in \mathbb{R}: |f(x) - f(x_0)| < \varepsilon\}$. Take $\delta = \min\{x_0 - a, b - x_0\}$, we see that $|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon$.
- "Only if". Assume f is continuous. Let A be open. For any $x_0 \in f^{-1}(A)$, there is $\varepsilon > 0$ such that $(f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subseteq A$. As f is continuous, there is $\delta > 0$ such that $|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon$, in other words $f((x_0 - \delta, x_0 + \delta)) \subseteq (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subseteq A$. Consequently $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(A)$. Therefore $f^{-1}(A)$ is open. □

Problem 27. Let $f(x): \mathbb{R} \mapsto \mathbb{R}$ be a real function satisfying $f(x) > 0$ for all $x \in \mathbb{R}$. Prove that for every closed interval $[a, b]$ with $a, b \in \mathbb{R}$, there is $\delta > 0$ such that $f(x) > \delta$ for all $x \in [a, b]$. Is the claim still true if one or both of a, b is infinity?

Proof. Forgot to put in the problem that $f(x)$ is continuous! When a, b are finite, we know that $m = \min_{x\in[a,b]} f(x)$. As $f(x) > 0$, $m \geqslant 0$. If $m = 0$, then there is $x_0 \in [a, b]$ such that $f(x_0) = 0$ contradiction. Therefore $m > 0$. Take $\delta = m/2$.

If one or both a, b is infinity, then the claim does not hold anymore. For example, if $b = \infty$, consider $f(x) = e^{-x}$. If $a = -\infty$, consider $f(x) = e^x$. □

Problem 28. (Cesaro average) Let $\{x_n\}$ be a real sequence. Set $y_n = (x_1 + \cdots + x_n)/n$. Show that if $x_n \longrightarrow a \in \mathbb{R}$, then $y_n \longrightarrow a \in \mathbb{R}$. What about the converse, that is does $y_n \longrightarrow a$ guarantees $x_n \longrightarrow a$?

Proof. Because $x_n \longrightarrow a \in \mathbb{R}$, there is $M \in \mathbb{R}$ such that $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. For any $\varepsilon > 0$, since $x_n \longrightarrow a$, there is $N_1 \in \mathbb{R}$ such that for all $n > N_1$, $|x_n - a| < \varepsilon/2$. Now take $N > \max\left\{\frac{2N_1(M + |a|)}{\varepsilon}, N_1\right\}$, for every $n > N$, we have

$$\begin{aligned} |y_n - a| &= \left|\frac{(x_1 - a) + (x_2 - a) + \cdots + (x_n - a)}{n}\right| \\ &\leqslant \frac{|x_1 - a| + |x_2 - a| + \cdots + |x_n - a|}{n} \\ &\leqslant \frac{|x_1 - a| + \cdots + |x_{N_1} - a|}{n} + \frac{|x_{N_1+1} - a| + \cdots + |x_n - a|}{n} \\ &\leqslant \frac{N_1(M + |a|)}{n} + \frac{(n - N_1)\varepsilon/2}{n} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \quad (60)$$

The converse is false. For example, take $x_n = (-1)^n$. Then $y_n \longrightarrow 0$. $\square$

Problem 29. Let $x_n > 0$ for all $n \in \mathbb{N}$. Show that

$$\liminf_{n \longrightarrow \infty} \frac{x_{n+1}}{x_n} \leqslant \liminf_{n \longrightarrow \infty} (x_n)^{1/n} \leqslant \limsup_{n \longrightarrow \infty} (x_n)^{1/n} \leqslant \limsup_{n \longrightarrow \infty} \frac{x_{n+1}}{x_n}. \tag{61}$$

Use this to prove that if $\lim \frac{x_{n+1}}{x_n}$ exists, so does $\lim (x_n)^{1/n}$. What about the converse?

Proof. The middle inequality is obvious.

- The left inequality $\liminf_{n \longrightarrow \infty} \frac{x_{n+1}}{x_n} \leqslant \liminf_{n \longrightarrow \infty} (x_n)^{1/n}$. For any $m > n$, we can write

$$(x_m)^{1/m} = x_n^{1/m} \left(\frac{x_{n+1}}{x_n} \cdots \frac{x_m}{x_{m-1}} \right)^{1/m} \geqslant x_n^{1/m} \left(\inf_{k \geqslant n} \frac{x_{k+1}}{x_k} \right)^{\frac{m-n}{n}}. \tag{62}$$

  As $x_n^{1/m} \longrightarrow 1$ and $\left(\inf_{k \geqslant n} \frac{x_{k+1}}{x_k} \right)^{\frac{m-n}{n}} \longrightarrow \inf_{k \geqslant n} \frac{x_{k+1}}{x_k}$ as $m \longrightarrow \infty$, we have

$$\liminf_{m \longrightarrow \infty} (x_m)^{1/m} \geqslant \liminf_{m \longrightarrow \infty} \left[x_n^{1/m} \left(\inf_{k \geqslant n} \frac{x_{k+1}}{x_k} \right)^{\frac{m-n}{n}} \right] = \lim_{m \to \infty} x_n^{1/m} \left(\inf_{k \geqslant n} \frac{x_{k+1}}{x_k} \right)^{\frac{m-n}{n}} = \inf_{k \geqslant n} \frac{x_{k+1}}{x_k}. \tag{63}$$

  Now take limit $n \longrightarrow \infty$, we conclude

$$\liminf_{m \longrightarrow \infty} (x_m)^{1/m} \geqslant \liminf_{n \longrightarrow \infty} \frac{x_{n+1}}{x_n}. \tag{64}$$

- The right inequality is proved similarly, using

$$(x_m)^{1/m} = x_n^{1/m} \left(\frac{x_{n+1}}{x_n} \cdots \frac{x_m}{x_{m-1}} \right)^{1/m} \leqslant x_n^{1/m} \left(\sup_{k \geqslant n} \frac{x_{k+1}}{x_k} \right)^{\frac{m-n}{n}} \tag{65}$$

  instead of (62).

If $\lim \frac{x_{n+1}}{x_n}$ exists, then $\liminf \frac{x_{n+1}}{x_n} = \limsup \frac{x_{n+1}}{x_n}$. Application of the squeeze theorem gives $\lim (x_n)^{1/n}$ exists.

The converse is false. For example take $x_n = 2 \pm (-1)^n$. Then $\lim x_n^{1/n} = 1$ but $\liminf \frac{x_{n+1}}{x_n} = 1/3$ while $\limsup \frac{x_{n+1}}{x_n} = 3$. $\square$