Math 314 Fall 2012 Final Review

Nov. 29, 2012 and Dec. 4, 2012

- Warning: This is **not** a complete list of materials covered by the exam. You still need to read the notes, review homeworks, and work on practice problems.

1. Concepts

1.1. Continuity.

- Definition.

  A function $f(x)\colon \mathbb{R} \mapsto \mathbb{R}$ is continuous at x_0 if

  1. $\lim_{x \longrightarrow x_0} f(x)$ exists;
  2. $f(x_0) = \lim_{x \longrightarrow x_0} f(x)$.

  Alternative formulation:

$$\lim_{h \longrightarrow 0} f(x_0 + h) = f(x_0). \tag{1}$$

- Show continuity at a point x_0.
  - If there is (a, b) such that $x_0 \in (a, b)$ and $f(x)$ on (a, b) is defined through combinations of elementary functions: Continuity of sum, difference, product, ratio, composition, inverse.
  - Otherwise: Definition. Possible methods of evaluating the limit $\lim_{x \longrightarrow x_0} f(x)$ (besides easy ones before the midterm)
    - Squeeze Theorem.

      Example 1. $f(x) = \begin{cases} |x|^{1/2} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous

      Proof. When $x \neq 0$, $|x|^{1/2}$, $\sin x$, $1/x$ all continuous so $f(x)$ continuous; At 0, use Squeeze:

$$-|x|^{1/2} \leqslant |x|^{1/2} \sin(1/x) \leqslant |x|^{1/2} \tag{2}$$

      So $\lim_{x \longrightarrow 0} f(x) = 0 = f(0)$. □

    - L'Hospital's Rule.

      Example 2. $f(x) = \begin{cases} x \ln x & x > 0 \\ 0 & x \leqslant 0 \end{cases}$ is continuous at $x_0 = 0$.

      Proof. By L'Hospital,

$$\lim_{x \longrightarrow 0} x \ln x = \lim_{x \longrightarrow 0} \frac{\ln x}{1/x} = \lim_{x \longrightarrow 0} \frac{1/x}{-1/x^2} = \lim_{x \longrightarrow 0} (-x) = 0 = f(0). \tag{3}$$

      □

      Note that we write $x \ln x$ as $\frac{\ln x}{1/x}$ instead of $\frac{x}{1/\ln x}$ because we want to get rid of $\ln x$ after taking derivative.

    - Taylor polynomial (equivalent to repeated application of L'Hospital's rule).

      Example 3. $f(x) = \begin{cases} \frac{1 - \cos x}{x^2} & x \neq 0 \\ 1/2 & x = 0 \end{cases}$ is continuous at 0.

Proof. We recall Taylor polynomial of $1-\cos x$:

$$1-\cos x = \frac{x^2}{2} - \frac{\sin \xi}{6} x^3 \tag{4}$$

with ξ between 0 and x. This gives, for $x \neq 0$,

$$\left| f(x) - \frac{1}{2} \right| = \left| \frac{\sin \xi}{6} x \right| \leqslant \frac{|x|}{6} \longrightarrow 0 \tag{5}$$

as $x \longrightarrow 0$. □

- Show discontinuity at a point x_0.
  - If $f(x_0)$ is known: Show $\lim_{x \longrightarrow x_0} f(x) \neq f(x_0)$; Or find $x_n \neq x_0$, $x_n \longrightarrow x_0$ but $f(x_n) \not\longrightarrow f(x_0)$.

Example 4. Prove that $f(x) = \begin{cases} \frac{x^2 - 5x + 4}{x^2 - 3x + 2} & x \neq 1 \\ 2 & x = 1 \end{cases}$ is not continuous at 1.

Proof. We have

$$\lim_{x \longrightarrow 1} f(x) = \lim_{x \longrightarrow 1} \frac{(x-4)(x-1)}{(x-2)(x-1)} = \lim_{x \longrightarrow 1} \frac{x-4}{x-2} = 3 \neq 2. \tag{6}$$

So f is not continuous at 1. □

  - If $f(x_0)$ is not known: Find $x_n \neq x_0$, $y_n \neq x_0$, $x_n \longrightarrow x_0$, $y_n \longrightarrow x_0$ but

$$\lim_{n \longrightarrow \infty} f(x_n) \neq \lim_{n \longrightarrow \infty} f(y_n). \tag{7}$$

Or show $\lim_{x \longrightarrow x_0+} f(x) \neq \lim_{x \longrightarrow x_0-} f(x)$ if the limits are easy to calculate.

Example 5. Prove that $f(x) = \begin{cases} \frac{1}{x-1} & x \neq 1 \\ c & x = 1 \end{cases}$ cannot be continuous at 1 no matter what c is.

Proof. Take $x_n > 1$, $y_n < 1$, $x_n \longrightarrow 1$, $y_n \longrightarrow 1$. Clearly

$$\lim_{n \longrightarrow \infty} \frac{1}{x_n - 1} = \infty, \qquad \lim_{n \longrightarrow \infty} \frac{1}{y_n - 1} = -\infty. \tag{8}$$

So $f(x)$ cannot be continuous at 1. □

- Popular counterexamples.

Example 6. The Heaviside function $H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leqslant 0 \end{cases}$ is continuous everywhere except at 0.

Example 7. $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous everywhere except at 0.

Note that the difference between the above two examples is that the left and right limits exist for the Heaviside function but not for the second function. So although both are discontinuous at 0, the former has much better behavior.

Example 8. The Dirichlet function $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$ is not continuous anywhere.

1.2. Differentiability.

- Definition.

$f(x)$ is differentiable at x_0 if

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{9}$$

exists and is finite. This value is denoted $f'(x_0)$.

Alternative formulation:

$$\lim_{h \longrightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0). \tag{10}$$

- Show differentiability at a point x_0.
  - If there is (a, b) such that $x_0 \in (a, b)$ and $f(x)$ on (a, b) is defined through combinations of elementary functions: Differentiability of sum, difference, product, ratio, composition, inverse.
  - Otherwise: Definition. Possible methods of evaluating the limit $\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$:
    - Squeeze Theorem of Limits.

Example 9. $f(x) = \begin{cases} |x|^{3/2} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable.

Proof. We see that

$$f(x) = \begin{cases} x^{3/2} \sin(1/x) & x > 0 \\ 0 & x = 0 \\ (-x)^{3/2} \sin(1/x) & x < 0 \end{cases} \tag{11}$$

so it is differentiable at all $x \neq 0$ because $x^{3/2}$, $\sin x$, $1/x$ are differentiable there. At $x_0 = 0$ we have

$$0 \leqslant \frac{f(x) - f(0)}{x - 0} = |x|^{1/2} \sin(1/x) \leqslant |x|^{1/2} \tag{12}$$

Application of Squeeze Theorem then gives $\lim_{x \longrightarrow 0} \frac{f(x) - f(0)}{x - 0} = 0$ and differentiability follows. □

- L'Hospital's Rule.

Example 10. $f(x) = \begin{cases} x^2 \ln x & x > 0 \\ 0 & x \leqslant 0 \end{cases}$ is differentiable.

Proof. Since x^2, $\ln x$ are differentiable for $x > 0$ so does $x^2 \ln x$. On the other hand 0 is differentiable for $x < 0$. So $f(x)$ is differentiable for $x \neq 0$. At 0 we have

$$\lim_{x \longrightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \longrightarrow 0} x \ln x = 0 \tag{13}$$

(The limit is proved using L'Hospital in Example 2.) □

1.3. Integrability.

- Definition.

Partition $P \longrightarrow$ Upper/Lower Riemann Sums $U(f, P), L(f, P) \longrightarrow$ Upper/Lower Riemann Integrals

$$U(f) = \inf_P U(f, P); \qquad L(f) = \sup_P L(f, P). \tag{14}$$

f is integrable if $U(f) = L(f)$ or equivalently

$$\inf_P U(f, P) = \sup_P L(f, P). \tag{15}$$

When $U(f) = L(f)$, the common value is denoted $\int_a^b f(x)\, dx$.

Remark 11. Note that

- The definition only makes sense when f is bounded and $a, b \in \mathbb{R}$. Otherwise $U(f), L(f)$ may be extended real numbers. Therefore if f is unbounded or one of a, b is $\pm\infty$, f is by definition **not integrable**, although we can still study its **improper integrability**.

- Prove integrability:
  - If f is combination of simpler functions:
    - Integrability of $f + g, c f, f g$.
    - Composition: f integrable, g continuous $\Longrightarrow$ $g \circ f$ integrable.
  - If f is continuous:

    If $f(x)$ is continuous on $[a, b]$, then it is integrable on $[a, b]$.

    Note that $f(x)$ continuous on (a, b) is not enough. For example $f(x) = 1/x$ on $(0, 1)$.
  - Definition.

    Take an arbitrary P, calculate $U(f,\ P)$, $L(f,\ P)$, then calculate $U(f)$, $L(f)$. Compare. Usually very difficult to do.
  - "Cauchy" –

    For each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon. \tag{16}$$

    Or equivalently,

    Choose P_n appropriately so that $U(f, P_n), L(f, P_n)$ are easy to calculate and $U(f, P_n) - L(f, P_n) \longrightarrow 0$.

    The first thing to try is to take $P_n = \left\{x_0 = a, x_1 = a + \frac{b-a}{n}, \ldots, x_n = b\right\}$.

    Example 12. Let $f(x)\colon [a, b] \mapsto \mathbb{R}$ be bounded and monotone, then $f(x)$ is integrable.

    Proof. Take $P_n = \left\{a, a + \frac{b-a}{n}, \ldots, b\right\}$ and show $U(f, P_n) - L(f, P_n) \longrightarrow 0$. The key observation is $\sum_{i=1}^n [\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f] = |f(b) - f(a)|$. □

1.4. Improper integrability.

- Definition:

$$\left\{ \begin{array}{c} f \text{ integrable on every } [c, d] \subset [a, b] \\ \displaystyle\lim_{c \to a+} \left(\lim_{d \to b-} \int_c^d f(x)\,\mathrm{d}x \right) \text{ exists and is finite} \end{array} \right\} \Rightarrow \int_a^b f(x)\,\mathrm{d}x = \lim_{c \to a+} \left(\lim_{d \to b-} \int_c^d f(x)\,\mathrm{d}x \right) \tag{17}$$

- Prove improper integrability/Calculate improper integrals:
  1. Identify for what $[c, d] \subseteq [a, b]$ is $f(x)$ integrable. If one can take $c = a$ or $d = b$, then the double limit can be simplified to a single limit.
  2. (If prove) Show existence and finiteness of the limit: We use $\lim_{c \longrightarrow b} \int_a^c f(x)\,\mathrm{d}x$ as example here.
     a) If $f(x) \geqslant 0$ or $\leqslant 0$ for all x, the integrals $\int_a^c f(x)\,\mathrm{d}x$ is monotone as a function of c, therefore it suffices to show boundedness (above if $\geqslant 0$, below if $\leqslant 0$); Usually it is shown through finding another function g which is improperly integrable on (a, b) and satisfies

$$\int_a^c f(x)\,\mathrm{d}x \leqslant (\geqslant \text{ if showing bounded below}) \int_a^c g(x)\,\mathrm{d}x. \tag{18}$$

b) General case, show $\int_a^c f(x)\,\mathrm{d}x$ is Cauchy (see Midterm Review).

3. (If calculate) Calculate $\int_c^d f(x)\,\mathrm{d}x$ and take appropriate limit(s).

Example 13. Show that $\frac{1}{1+x^{4/3}}$ is improperly integrable over $(0,\infty)$.

Proof. Since $\frac{1}{1+x^{4/3}}$ is continuous on $[0,c]$ for all c, we only need to show the existence and finiteness of the limit

$$\lim_{c\longrightarrow\infty}\int_0^c \frac{\mathrm{d}x}{1+x^{4/3}}. \tag{19}$$

Notice that $\frac{1}{1+x^{4/3}}>0$, all we need to show is $\int_0^c \frac{\mathrm{d}x}{1+x^{4/3}}$ is bounded above. For any $c\leqslant 1$ we have

$$\int_0^c \frac{\mathrm{d}x}{1+x^{4/3}}\leqslant\int_0^1 \frac{\mathrm{d}x}{1+x^{4/3}}\leqslant\int_0^1 \mathrm{d}x=1; \tag{20}$$

On the other hand for $c>1$, we have

$$\int_0^c \frac{\mathrm{d}x}{1+x^{4/3}}=\int_0^1 \frac{\mathrm{d}x}{1+x^{4/3}}+\int_1^c \frac{\mathrm{d}x}{1+x^{4/3}}<\int_0^1 \mathrm{d}x+\int_1^\infty \frac{\mathrm{d}x}{x^{4/3}}=4. \tag{21}$$

Therefore the limit exists and is finite and $\frac{1}{1+x^{4/3}}$ is improperly integrable over $(0,\infty)$. □

1.5. Infinite series.

- Convergence:

$$\sum_{n=1}^\infty a_n \text{ converges} \Longleftrightarrow \left\{S_n=\sum_{k=1}^n a_n\right\} \text{ converges.} \tag{22}$$

- Divergence to $\pm\infty$:

$$\sum_{n=1}^\infty a_n=\infty(-\infty)\Longleftrightarrow \lim_{n\longrightarrow\infty} S_n=\infty(-\infty). \tag{23}$$

- Show convergence:
  - $a_n\geqslant 0$ ($\leqslant 0$): Find upper bound (lower bound);
  - General case:
    - If the limit is known or the formula for S_n is easy to obtain: Definition.
    - Otherwise:
      - Cauchy:

$$\sum_{n=1}^\infty a_n \text{ converges} \Longleftrightarrow \left\{\forall\varepsilon>0, \exists N\in\mathbb{N}, \forall m,n>N, \left|\sum_{k=n+1}^m a_n\right|<\varepsilon\right\} \tag{24}$$

      - Dominance:

$$\text{Find } \sum_{n=1}^\infty b_n \text{ convergent and } N_0\in\mathbb{N} \text{ and } c>0, \quad c\,b_n\geqslant|a_n| \text{ for all } n>N_0. \tag{25}$$

- Show divergence to $\pm\infty$: Definition.
- Show divergence (including divergence to $\pm\infty$):
  - $a_n\geqslant 0$ ($\leqslant 0$): Show unboundedness.
  - General case:
    - Show $a_n\not\longrightarrow 0$;
    - Find b_n such that $a_n\geqslant|b_n|$ for all $n>N_0\in\mathbb{N}$ and $\sum_{n=1}^\infty b_n$ does not converge.

– Definition: Show $\lim_{n\longrightarrow\infty} S_n$ does not exist where $S_n = \sum_{k=1}^{n} a_k$.

2. Theorems

2.1. Continuity.

- Intermediate Value Theorem:

  Let $f(x)$ be continuous on the closed interval $[a, b]$. Then for every $s \in [f(a), f(b)]$ (or $[f(b), f(a)]$ if $f(b) \leqslant f(a)$), there is $\xi \in [a, b]$ such that $f(\xi) = s$.

  Note that $f(x)$ must be continuous at the end points a, b.

  Example 14. Prove that $f(x) = x^3 + 2x - 10 = 0$ has exactly one solution in $\mathbb{R}$.

  Proof. First we show that it has at least one solution using intermediate value theorem. To do this we need to find $a, b \in \mathbb{R}$ such that $f(a)$, $f(b)$ take different signs. It is easy to see that this can be done by taking $|x|$ large. For example take $a = -10, b = 10$.

  To show that this is the only solution, we show $f(x)$ is strictly increasing by showing $f'(x) > 0$. Calculate $f'(x) = 3x^2 + 2 \geqslant 2 > 0$. □

- Boundedness, Maximum, Minimum.

  Let $f(x)$ be continuous on $[a, b]$ for $a, b \in \mathbb{R}$. Then there are $x_{\max}, x_{\min} \in [a, b]$ such that

$$f(x_{\max}) \geqslant f(x), \qquad f(x_{\min}) \leqslant f(x) \qquad \forall x \in [a, b]. \tag{26}$$

  Of course this gives the boundedness of f:

$$f(x) \in [f(x_{\min}), f(x_{\max})]. \tag{27}$$

2.2. Calculation of derivatives.

- Arithmetics.

$$(f+g)' = f' + g'; \qquad (c\,f)' = c\,f'; \qquad (f\,g)' = f'\,g + f\,g'; \qquad \left(\frac{f}{g}\right)' = \frac{f'\,g - g'\,f}{g^2}. \tag{28}$$

  To remember the last one, use $(x^{-1})' = -x^{-2}$.

- Chain Rule.

$$g(f(x))' = g'(f(x))\,f'(x). \tag{29}$$

  Note that $g'(f(x))$ means take the function $g'(x)$ and evaluate it at the point $f(x)$.

- Inverse Function. If g is the inverse of f, then

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))}. \tag{30}$$

  The tricky issue here is how to obtain a formula for $f'(g(y))$.

  Example 15. Let $f(x) = e^x + x - \cos x$. Prove that it has an inverse function g which is differentiable. Calculate $g'(0)$.

  Solution. We have

$$f'(x) = e^x + 1 + \sin x \geqslant e^x > 0 \tag{31}$$

  for all $x \in \mathbb{R}$ so f is strictly increasing and thus have an inverse function g. Since f is differentiable with $f' > 0$, g is differentiable. To calculate $g'(0)$ we write

$$g'(0) = \frac{1}{f'(x)} \tag{32}$$

where x is such that $f(x)=0$. Clearly $x=0$ so

$$g'(0)=\frac{1}{e^0+1+\sin 0}=\frac{1}{2}. \tag{33}$$

2.3. Understanding functions through derivatives.

- f is differentiable at x_0 then it is continuous at x_0.
- Monotonicity, maximum, minimum.

We have

$$f'(x)\geqslant(\leqslant)0 \iff \text{increasing (decreasing)}; \tag{34}$$
$$f'(x)>(<)\,0 \Longrightarrow \text{strictly increasing (decreasing)}; \tag{35}$$
$$f'(x)=0 \iff \text{constant}; \tag{36}$$
$$\left.\begin{array}{r} f(x_0) \text{ is local maximum/minimum} \\ f'(x_0) \text{ exists} \end{array}\right\} \Longrightarrow f'(x_0)=0. \tag{37}$$

Example 16. Find all local maximizers of $f(x)=x^3-3\,x+7$.
Solution. Since $f(x)$ is differentiable, its maximizer(s) solve $f'(x)=0$ which is

$$3\,x^2-3=0. \tag{38}$$

Therefore possible candidates are $x=1,-1$. Now we study the sign of $f'(x)$ in $(-\infty,-1)$, $(-1,1)$, $(1,\infty)$. We see that $f'(x)$ is positive, negative, positive respectively. This means $f(x)$ is increasing when $x<-1$, decreasing when $x\in(-1,1)$ and increasing when $x>1$. Therefore -1 is the maximizer and 1 is the minimizer.

- Mean Value Theorem and Cauchy's Extended Mean Value Theorem.

(Mean Value Theorem) Let f be **continuous on** $[a,b]$ and **differentiable on** (a,b). Then there is a point $\xi\in(a,b)$ such that

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}. \tag{39}$$

(Cauchy's Extended Mean Value Theorem) Let f,g be continuous over $[a,b]$ and differentiable over (a,b). Then there is $\xi\in(a,b)$ such that

$$\frac{f(a)-f(b)}{g(a)-g(b)}=\frac{f'(\xi)}{g'(\xi)}. \tag{40}$$

The Extended Mean Value Theorem is very powerful when $g(x)$ is chosen cleverly.

Example 17. Prove that if $f''(x)$ exists at $x=x_0$, then

$$\lim_{h\longrightarrow 0}\frac{f(x_0+h)+f(x_0-h)-2\,f(x_0)}{h^2}=f''(x_0). \tag{41}$$

Proof. Take $F(h):=f(x_0+h)+f(x_0-h)$, $G(h)=h^2$ and apply Cauchy's Extended Mean Value Theorem:

$$\frac{f(x_0+h)+f(x_0-h)-2\,f(x_0)}{h^2}=\frac{F(h)-F(0)}{G(h)-G(0)}=\frac{F'(\xi)}{G'(\xi)}=\frac{f'(x_0+\xi)-f'(x_0-\xi)}{2\,\xi} \tag{42}$$

where $\xi\in(0,h)$. Taking limit $h\longrightarrow 0$ we have $\xi\longrightarrow 0$ and

$$\frac{f'(x_0+\xi)-f'(x_0-\xi)}{2\,\xi}\longrightarrow f''(x_0) \tag{43}$$

as we have shown in Homework 5 Problem 1. □

- L'Hospital's Rule. $x_0 \in \mathbb{R} \cup \{\infty, -\infty\}$.

$$\left\{ \begin{array}{l} f, g \text{ differentiable on } (a, x_0) \cup (x_0, b) \\ \frac{\lim_{x \longrightarrow x_0} f(x)}{\lim_{x \longrightarrow x_0} g(x)} \text{ undetermined: } \frac{0}{0}, \frac{\pm\infty}{\pm\infty} \end{array} \right\} \Longrightarrow \lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \longrightarrow x_0} \frac{f'(x)}{g'(x)} \tag{44}$$

if the latter exists.

Example 18. Usually there may be more than one way to write a function into a ratio. Choose the one that gets simpler after differentiation. For example, if we study $\lim_{x \longrightarrow \infty} x\, e^{-x^2}$ we should write

$$x\, e^{-x^2} = \frac{x}{e^{x^2}} \tag{45}$$

but not

$$x\, e^{-x^2} = \frac{e^{-x^2}}{1/x}. \tag{46}$$

- Taylor polynomial with Lagrange form of remainder.

Let f be such that $f^{(k)}(x)$ exists on (a, b) for $k = 0, \ldots, n + 1$. Then $f(x)$ can be written as a the sum of a polynomial and a remainder term:

$$f(x) = T_n(x) + R_n(x) \tag{47}$$

where its Taylor polynomial of degree n reads

$$T_n(x) = f(x_0) + f'(x_0)\,(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n \tag{48}$$

and its Lagrange form of remainder reads

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x - x_0)^{n+1} \tag{49}$$

where ξ is between x, x_0.

Note that both $T_n(x)$ changes if x_0 changes, while $R_n(x)$ changes if either x_0 or x changes since $\xi = \xi(x, x_0)$.

Example 19. Calculate Taylor polynomial of $f(x) = \arctan x$ at $x_0 = 0$ to degree 2 with Lagrange form of remainder.
Solution. We need $f(0)$, $f'(0)$, $f''(0)$, $f'''(\xi)$.

$$f(x) = \arctan x \Longrightarrow f(0) = 0. \tag{50}$$

$$f'(x) = \frac{1}{1 + x^2} \Longrightarrow f'(0) = 1. \tag{51}$$

$$f''(x) = \frac{-2\,x}{(1 + x^2)^2} \Longrightarrow f''(0) = 0. \tag{52}$$

$$f'''(x) = \frac{-2\,(1 + x^2)^2 - (-2\,x)\,(4\,x)\,(1 + x^2)}{(1 + x^2)^4} = \frac{-2 + 6\,x^2}{(1 + x^2)^3} \Longrightarrow f'''(\xi) = \frac{-2 + 6\,\xi^2}{(1 + \xi^2)^3}. \tag{53}$$

Therefore

$$T_2(x) = x, \qquad R_2(x) = \frac{-2 + 6\,\xi^2}{6\,(1 + \xi^2)^3}\,x^3. \tag{54}$$

and

$$\arctan x = x + \frac{-2 + 6\,\xi^2}{6\,(1 + \xi^2)^3}\,x^3 \tag{55}$$

for some ξ between 0 and x.

2.4. Integrability.

- f is integrable on $[a, b] \Longleftrightarrow f$ is integrable on $[a, c]$ and $[c, b]$ for every $c \in (a, b)$. Also true for improper integrals.

- f is integrable then so is $|f|$. Note that this is not true for improper integrals.

2.5. Manipulation of integrals.

- Useful relations:

$$f \leqslant g \Longrightarrow \int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x. \tag{56}$$

$$\left|\int_a^b f\,\mathrm{d}x\right| \leqslant \int_a^b |f|\,\mathrm{d}x. \tag{57}$$

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x. \tag{58}$$

- Fundamental Theorem of Calculus Version 1:

$$\left\{\begin{array}{c} f \text{ integrable on } [a,b] \\ F \text{ antiderivative on } (a,b) \\ F \text{ continuous on } [a,b] \end{array}\right\} \Longrightarrow \int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) \tag{59}$$

- Fundamental Theorem of Calculus Version 2:

$$f \text{ integrable on } [a,b] \Longrightarrow G(x) := \int_a^x f(t)\,\mathrm{d}t \text{ continuous on } [a,b] \tag{60}$$

$$\left\{\begin{array}{c} f \text{ integrable on } [a,b] \\ f \text{ continuous at } x_0 \in (a,b) \end{array}\right\} \Longrightarrow G'(x_0) = f(x_0). \tag{61}$$

The most important thing to keep in mind is

$$G(x) := \int_a^x f(t)\,\mathrm{d}t \Longrightarrow \int_a^{h(x)} f(t)\,\mathrm{d}t = G(h(x)). \tag{62}$$

2.6. Infinite Series.

- $\sum_{n=1}^{\infty} |a_n|$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges.
- Tests for non-negative series:
  - Ratio test:

$$\limsup_{n\longrightarrow\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Longrightarrow \text{converge}; \qquad \liminf_{n\longrightarrow\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \Longrightarrow \text{diverge} \tag{63}$$

  - Root test:

$$\limsup_{n\longrightarrow\infty} |a_n|^{1/n} < 1 \Longrightarrow \text{converge}; \qquad \liminf_{n\longrightarrow\infty} |a_n|^{1/n} > 1 \Longrightarrow \text{diverge}. \tag{64}$$

Example 20. Study the convergence/divergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$ for $x \in \mathbb{R}$.
Solution. We apply the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{n\,|x|}{n+1} \longrightarrow |x|. \tag{65}$$

Therefore the series converges when $|x| < 1$ and diverges when $|x| > 1$. At $|x| = 1$,

- $x = 1$: In this case the series is $\sum_{n=1}^{\infty} \frac{1}{n}$ and we know it diverges;
- $x = -1$: In this case the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ and we know it converges.

Summarizing, the series converges when $-1 \leqslant x < 1$ and diverges elsewhere.

Remark 21. Note that in general, the "tests" are only useful when a_n's are explicitly given. For questions dealing with general a_n, one should try dominance ((25)) or Cauchy/definition (in that order).

- Useful facts:
  - Geometric series.

$$\sum_{n=1}^{\infty} r^{n-1} = \begin{cases} \dfrac{1}{1-r} & |r|<1 \\ \infty & r \geqslant 1 \\ \text{does not exist} & r \leqslant -1 \end{cases}. \tag{66}$$

  - Harmonic series and generalizations.

$$\sum_{n=1}^{\infty} n^{-a} \begin{cases} \text{converges} & a>1 \\ =\infty & a \leqslant 1 \end{cases}. \tag{67}$$

  - Alternating series. $a_n \geqslant 0$, a_n decreasing, $a_n \longrightarrow 0$. Then

$$\sum_{n=1}^{\infty} (-1)^n a_n \tag{68}$$

converges. In particular $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

3. TRICKS

3.1. The d operator.

- A more efficient way of writing down integration by parts and change of variables.

For a single variable function $f(x)$ we simply have

$$\mathrm{d} f = f'(x)\, \mathrm{d}x \tag{69}$$

thus

$$\mathrm{d}(\cos x) = (-\sin x)\, \mathrm{d}x; \qquad \mathrm{d}(\arctan x) = \frac{\mathrm{d}x}{1+x^2} \tag{70}$$

etc.

The integration by parts formula now becomes

$$\int_a^b u\, \mathrm{d}v = u(b)\, v(b) - u(a)\, v(a) - \int_a^b v\, \mathrm{d}u. \tag{71}$$

And the change of variables formula now can be written in a form that is often more natural:

$$\int_a^b f(x)\, \mathrm{d}x \overset{x=u(y)}{=\!=\!=} \int_c^d f(u(y))\, \mathrm{d}u(y) = \int_c^d f(u(y))\, u'(y)\, \mathrm{d}y. \tag{72}$$

where c, d are such that $u(c) = a, u(d) = b$.

Example 22. Calculate $\int_1^e x \ln x\, \mathrm{d}x$.

We have

$$\begin{aligned} \int_1^e x \ln x\, \mathrm{d}x &= \int_1^e \ln x\, \mathrm{d}\left(\frac{x^2}{2}\right) \\ &= \left[\frac{x^2}{2} \ln x\right]_1^e - \int_1^e \frac{x^2}{2}\, \mathrm{d}(\ln x) \\ &= \frac{e^2}{2} - \int_1^e \frac{x}{2}\, \mathrm{d}x \\ &= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4}. \end{aligned} \tag{73}$$

Example 23. Calculate

$$\int_1^4 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\, \mathrm{d}x \tag{74}$$

Solution. It's natural to set $x = y^2$. Then we have

$$\begin{aligned}\int_1^4 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x &= \int_1^2 \frac{\sqrt{1+y}}{y}\,\mathrm{d}(y^2)\\ &= 2\int_1^2 \sqrt{1+y}\,\mathrm{d}y\\ &= \frac{4}{3}(1+y)^{3/2}\Big|_{y=1}^{y=2}\\ &= \frac{4}{3}\left[3^{3/2}-2^{3/2}\right].\end{aligned}$$

Example 24. Calculate

$$\int_0^1 \frac{e^{-\sqrt{x}}}{2\sqrt{x}}\,\mathrm{d}x \tag{75}$$

Solution. Again it's natural to set $x = u(y) = y^2$. This gives

$$\begin{aligned}\int_0^1 \frac{e^{-\sqrt{x}}}{2\sqrt{x}}\,\mathrm{d}x &= \int_0^1 \frac{e^{-y}}{2\,y}\,\mathrm{d}(y^2)\\ &= \int_0^1 e^{-y}\,\mathrm{d}y\\ &= 1-e^{-1}.\end{aligned} \tag{76}$$

3.2. Abel's summation formula.

Example 25. Let a_n be such that $\lim_{n\longrightarrow\infty} n\,a_n$ exists and is finite, $\sum_{n=1}^{\infty} n\,(a_n - a_{n-1})$ converges. Prove that $\sum_{n=1}^{\infty} a_n$ converges.

Proof. We show that $\sum_{n=1}^{\infty} a_n$ is Cauchy. For any $m > n$ we have

$$\begin{aligned}\sum_{k=n+1}^{m} a_k &= \sum_{k=n+1}^{m} [k-(k-1)]\,a_k\\ &= \sum_{k=n+1}^{m} k\,a_k - \sum_{k=n+1}^{m}(k-1)\,a_k\\ &= \sum_{k=n+1}^{m} k\,a_k - \sum_{k=n}^{m-1} k\,a_{k+1}\\ &= m\,a_m - n\,a_{n+1} + \sum_{k=n+1}^{m-1} k\,(a_k - a_{k+1}).\end{aligned} \tag{77}$$

Note that if $n\,a_n$ converges to $s\in\mathbb{R}$, so does $n\,a_{n+1} = \frac{n+1}{n}[(n+1)\,a_{n+1}]$. Now for any $\varepsilon > 0$, take N_1, N_2, $N_3\in\mathbb{N}$ such that for all $n > N_1$, $|n\,a_n - s| < \varepsilon/3$; For all $n > N_2$, $|n\,a_{n+1} - s| < \varepsilon/3$; For all $m > n > N_3$,

$$\left|\sum_{k=n+1}^{m-1} k\,(a_k - a_{k+1})\right| < \varepsilon/3. \tag{78}$$

Take $N = \max\{N_1, N_2, N_3\}$. Now for all $m > n > N$, we have

$$\begin{aligned}\left|m\,a_m - n\,a_{n+1} + \sum_{k=n+1}^{m-1} k\,(a_k - a_{k+1})\right| &\leqslant |(m\,a_m - s) - (n\,a_{n+1} - s)|\\ &\quad + \left|\sum_{k=n+1}^{m-1} k\,(a_k - a_{k+1})\right|\\ &\leqslant |m\,a_m - s| + |n\,a_{n+1} - s| + \left|\sum_{k=n+1}^{m-1} k\,(a_k - a_{k+1})\right|\\ &< \varepsilon.\end{aligned} \tag{79}$$

Therefore $\sum_{n=1}^{\infty} a_n$ is Cauchy and the proof ends. □

Example 26. Let $\alpha > 0$. Then $\sum_{n=1}^{\infty} a_n$ converges $\Longrightarrow \sum_{n=1}^{\infty} n^{-\alpha} a_n$ converges.

Proof. Since $\sum_{n=1}^{\infty} a_n$ converges, we have $S_n = \sum_{k=1}^{n} a_n$ converges (by definition) and therefore bounded. Let M be such that $|S_n| \leqslant M$ for all $n \in \mathbb{N}$. Now write

$$\begin{aligned}
\sum_{k=n+1}^{m} k^{-\alpha} a_k &= \sum_{k=n+1}^{m} k^{-\alpha} (S_k - S_{k-1}) \\
&= \sum_{k=n+1}^{m} k^{-\alpha} S_k - \sum_{k=n+1}^{m} k^{-\alpha} S_{k-1} \\
&\overset{l=k-1 \text{ in 2nd sum}}{=\!=\!=\!=} \sum_{k=n+1}^{m} k^{-\alpha} S_k - \sum_{l=n}^{m-1} (l+1)^{-\alpha} S_l \\
&= \sum_{k=n+1}^{m} k^{-\alpha} S_k - \sum_{k=n}^{m-1} (k+1)^{-\alpha} S_k \\
&= m^{-\alpha} S_m + \sum_{k=n+1}^{m-1} [k^{-\alpha} - (k+1)^{-\alpha}] S_k - (n+1)^{-\alpha} S_n. \qquad (80)
\end{aligned}$$

Therefore

$$\left| \sum_{k=n+1}^{m} k^{-\alpha} a_k \right| \leqslant 2 (n+1)^{-\alpha} M. \qquad (81)$$

Now it's easy to show that $\sum_{n=1}^{\infty} n^{-\alpha} a_n$ is Cauchy and thus converges. □

Remark 27. In fact this is a special case of Problem 5 a) of HW7.

3.3. Telescoping.

Example 28. (USTC) $a_n > 0$. Recall $S_n = \sum_{k=1}^{n} a_n$. Then $\sum_{n=1}^{\infty} \frac{a_n}{S_n^2}$ converges.

Proof. Write (for $n \geqslant 2$)

$$\frac{a_n}{S_n^2} = \frac{S_n - S_{n-1}}{S_n^2} < \frac{S_n - S_{n-1}}{S_{n-1} S_n} = \frac{1}{S_{n-1}} - \frac{1}{S_n}. \qquad (82)$$

This gives

$$\sum_{k=1}^{n} \frac{a_k}{S_k^2} < \frac{1}{a_1} + \sum_{k=2}^{n} \left[\frac{1}{S_{k-1}} - \frac{1}{S_k} \right] = \frac{2}{a_1} - \frac{1}{S_n} < \frac{2}{a_1}. \qquad (83)$$

Since $\frac{a_n}{S_n^2} > 0$, this upper bound implies convergence. □

3.4. Construction of counterexamples.

Example 29. HW4 Problem 4 a).