Math 314 Fall 2012 Final Practice

- You should also
  - review homework problems.
  - try the 2011 final (and you should feel most of its problems are easy).
- Most problems in the final will be at the "Basic" and "Intermediate" levels (First 36 problems).

Basic

Problem 1. Let

$$f(x) = \begin{cases} -1 & x \leqslant -1 \\ a\,x^2 + b\,x + c & |x| < 1, x \neq 0 \\ 0 & x = 0 \\ 1 & x \geqslant 1 \end{cases}. \tag{1}$$

Find $a, b, c \in \mathbb{R}$ such that $f(x)$ is continuous at every x.

Problem 2. Calculate the derivatives of the following functions.

$$f_1(x) = \left(\frac{1+x^2}{1-x^2} \right)^3; \qquad f_2(x) = \sqrt{1+x+x^2}; \qquad f_3(x) = \exp\,[x \ln x]. \tag{2}$$

Problem 3. Calculate the following limits.

$$\lim_{x \longrightarrow 0} \frac{1-\cos^2 x}{\sqrt{1+x^2}-1}; \qquad \lim_{x \longrightarrow 0} \frac{e^x - e^{-x} - 2\,x}{x - \sin x}; \qquad \lim_{x \longrightarrow \infty} \frac{\pi - \arctan x}{\sin\,(1/x)}. \tag{3}$$

Problem 4. Calculate Taylor polynomial to degree 2 with Lagrange form of remainder.

$$f(x) = x \sin\,(\ln x); \qquad x_0 = 1. \tag{4}$$

Problem 5. Let $f(x) = 2\,x - \sin\,x$ defined on $\mathbb{R}$. Prove that its inverse function g exists and is differentiable. Then calculate $g'(0)$, $g'(\pi - 1)$.

Problem 6. Which of the following functions is/are differentiable at $x_0 = 0$? Justify your answers

$$f_1(x) = \begin{cases} x+2 & x > 0 \\ x-2 & x \leqslant 0 \end{cases}; \qquad f_2(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}; \qquad f_3(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}. \tag{5}$$

Problem 7. Let $f(x): \mathbb{R} \mapsto \mathbb{R}$ be continuous and $x_0 \in E$. Define $F(x) := \begin{cases} \frac{f(x) - f(x_0)}{x - x_0} & x \neq x_0 \\ c & x = x_0 \end{cases}$. Prove that f is differentiable at x_0 if and only if there is $c \in \mathbb{R}$ such that $F(x)$ is continuous for all $x \in \mathbb{R}$.

Problem 8. Calculate the following integrals:

$$I_1 = \int_e^{e^2} \frac{\mathrm{d}x}{x\,(\ln x)^4}; \qquad I_2 = \int_0^4 e^{-\sqrt{x}}\,\mathrm{d}x; \qquad I_3 = \int_1^e x^3 \ln x\,\mathrm{d}x \tag{6}$$

Problem 9. Prove that the following improper integrals exist and calculate their values:

$$J_1 = \int_0^\infty e^{-2x}\cos(3\,x)\,\mathrm{d}x; \qquad J_2 = \int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}}; \qquad J_3 = \int_0^1 (\ln x)^2\,\mathrm{d}x \tag{7}$$

Problem 10. Prove by definition that $f(x) = \left\{ \begin{array}{ll} 1 & x=0 \\ 0 & x \neq 0 \end{array} \right.$ is integrable over $[-1, 1]$ and find the value of $\int_{-1}^1 f(x)\,\mathrm{d}x$.

Problem 11. (USTC) Is the following calculation correct? Justify your answer.

$$\int_0^\pi \cos^2 x\,\mathrm{d}x = \int_0^0 \frac{\mathrm{d}t}{(1+t^2)^2} = 0 \tag{8}$$

where the change of variable is $t = \tan x$.

Problem 12. Let $F(x) := \int_{\sin x}^{x^2+2} e^t\,\mathrm{d}t$. Calculate $F'(x)$ and $F''(x)$.

Problem 13. Prove the convergence/divergence of (can use convergence/divergence of $\sum n^a$ and $\sum r^n$).

$$\sum_{n=1}^\infty \frac{2^n+n}{3^n+5\,n+4}, \qquad \sum_{n=1}^\infty \frac{n^2+n}{n^5-4}, \qquad \sum_{n=1}^\infty \frac{\sqrt{n^2+1}}{(n^{1/3}+19)^5}. \tag{9}$$

Problem 14. Prove that $\sum_{n=1}^\infty \frac{1}{n\,(n+3)}$ converges and find its sum.

Problem 15. Prove: If $\sum_{n=1}^\infty a_n^2$ converges then $\sum_{n=1}^\infty \frac{a_n}{n}$ converges. (Hint: $\frac{a^2+b^2}{2} \geqslant a\,b$)

Problem 16. Study the convergence/divergence of

$$\sum_{n=1}^\infty \frac{1}{n^{(1+1/n)}}. \tag{10}$$

Problem 17. Prove that

$$\sum_{n=1}^\infty \frac{1}{2\,n+1} = \infty. \tag{11}$$

You can use the divergence of $\sum_{n=1}^\infty \frac{1}{n}$.

Problem 18. Consider $\sum_{n=1}^\infty n\,r^n$. Identify the values of $r \in \mathbb{R}$ such that it is convergent. Justify your answer. You can use the fact that $\lim_{n\longrightarrow\infty} n\,r^n = 0$ when $|r| < 1$.

Intermediate

Problem 19. Let f, g be continuous at $x_0 \in \mathbb{R}$. Then so are

$$F(x) := \max\{f(x), g(x)\}, \qquad G(x) := \min\{f(x), g(x)\}. \tag{12}$$

Problem 20. Prove the following.

a) There is **exactly one** $x \in (0,1)$ such that

$$x^{1/2} e^x = 1. \tag{13}$$

b) There are infinitely many $x \in \mathbb{R}$ satisfying

$$x \sin x = 1. \tag{14}$$

Problem 21. Let $f(x)$ be differentiable at x_0 with derivative $f'(x_0) = 3$. Calculate

$$\lim_{n \longrightarrow \infty} (3n^2 + 2n - 1)\left[f\left(x_0 + \frac{2}{n^2}\right) - f(x_0)\right] \tag{15}$$

Problem 22. Let f, g be differentiable on (a, b) and continuous on $[a, b]$. Further assume $f(a) = g(b)$, $f(b) = g(a)$. Prove that there is $\xi \in (a, b)$ such that $f'(\xi) = -g'(\xi)$.

Problem 23. Prove the following inequalities

a) $|\cos x - \cos y| \leqslant |x - y|$ for all $x, y \in \mathbb{R}$;

b) $|\arctan x - \arctan y| \leqslant |x - y|$ for all $x, y \in \mathbb{R}$;

c) $\frac{a-b}{a} < \ln \frac{a}{b} < \frac{a-b}{b}$, $0 < b < a$.

Problem 24.

a) Let $a \in (0,1)$. Prove that

$$\lim_{n \longrightarrow \infty} [(n+1)^a - n^a] = 0. \tag{16}$$

You can use $(x^a)' = a\, x^{a-1}$.

b) Prove that

$$\lim_{n \longrightarrow \infty} \left[\sin\left((n+1)^{1/3}\right) - \sin\left(n^{1/3}\right)\right] = 0. \tag{17}$$

Problem 25. (USTC) Let f be differentiable on $\mathbb{R}$, $f(0) = 0$ and $f'(x)$ is strictly increasing. Prove that $\frac{f(x)}{x}$ is strictly increasing on $(0, \infty)$.

Problem 26. Let $f(x)$ be differentiable on $(-\infty, 0)$ and $(0, \infty)$. Assume that

$$\lim_{x \longrightarrow 0-} f'(x) = A, \qquad \lim_{x \longrightarrow 0+} f'(x) = B. \tag{18}$$

Prove that if $A \neq B$ then $f(x)$ is not differentiable at $x = 0$.

Problem 27. Let $a > 1$. Assume $f(x)$ satisfies $|f(x) - f(y)| \leqslant |x - y|^a$ for all $x, y \in \mathbb{R}$. Prove that f is constant.

Problem 28. (USTC) Let f, g be differentiable on $[a, \infty)$, and $|f'(x)| \leqslant g'(x)$ for all $x \in [a, \infty)$. Prove that

$$|f(x) - f(a)| \leqslant g(x) - g(a) \tag{19}$$

for all $x > a$. (Hint: Cauchy's generalized mean value theorem.)

Problem 29. Let f be continuous and g be integrable on $[a, b]$. Further assume that $g(x)$ doesn't change sign in $[a, b]$. Prove that there is $\xi \in [a, b]$ such that

$$\int_a^b f(x)\, g(x)\, \mathrm{d}x = f(\xi) \int_a^b g(x)\, \mathrm{d}x. \tag{20}$$

Does the conclusion still hold if we drop "$g(x)$ doesn't change sign in $[a, b]$"?

Problem 30. Prove the following inequalities:

a) $\int_0^1 e^{-x^2}\, \mathrm{d}x > \int_1^2 e^{-x^2}\, \mathrm{d}x$;

b) $\int_0^{\pi/2} \frac{\sin x}{x}\, \mathrm{d}x > \int_0^{\pi/2} \frac{\sin^2 x}{x^2}\, \mathrm{d}x$;

Problem 31. (USTC) Prove

$$\int_0^{2\pi} \left[\int_x^{2\pi} \frac{\sin t}{t}\, \mathrm{d}t \right] \mathrm{d}x = 0. \tag{21}$$

(Hint: Set $u(x) = \int_x^{2\pi} \frac{\sin t}{t}\, \mathrm{d}t$ then integrate by parts)

Problem 32. Let f be continuous on $\mathbb{R}$. Let $a, b \in \mathbb{R}, a < b$. Then

$$\lim_{h \longrightarrow 0} \int_a^b \frac{f(x+h) - f(x)}{h}\, \mathrm{d}x = f(b) - f(a). \tag{22}$$

Problem 33. (USTC) Let f be integrable. Prove that

$$\int_0^\pi x\, f(\sin x)\, \mathrm{d}x = \frac{\pi}{2} \int_0^\pi f(\sin x)\, \mathrm{d}x. \tag{23}$$

(Hint: Change of variable: $t = \pi - x$.)

Problem 34. Apply Ratio/Root tests to determine the convergence/divergence of the following series (You need to decide which one is more convenient to use).

$$\sum_{n=1}^\infty \frac{1}{2^n} (1 + 1/n)^{n^2}; \qquad \sum_{n=1}^\infty (n!)\, x^n; \qquad \sum_{n=1}^\infty \frac{(n!)}{n^n} x^n. \tag{24}$$

You can use the fact $(1 + 1/n)^n \longrightarrow e$, and the Stirling's formula

$$\lim_{n \longrightarrow \infty} \frac{n!}{\sqrt{2\, \pi\, n}\, (n/e)^n} = 1 \tag{25}$$

without proof.

Problem 35. $a_n \geqslant 0$, $\sum_{n=1}^\infty a_n$ converges. Prove that $\sum_{n=1}^\infty \sqrt{a_n\, a_{n+1}}$ converges. On the other hand, if a_n furthermore is decreasing, then $\sum_{n=1}^\infty \sqrt{a_n\, a_{n+1}}$ converges $\Longrightarrow \sum_{n=1}^\infty a_n$ converges. Any example of if a_n is not decreasing then not true? (Take $a_n = 0$ for all n even)

Problem 36. Let $a_n \geqslant 0$. Prove that $\sum_{n=1}^\infty a_n$ converges then $\sum_{n=1}^\infty a_n^2$ converges. Is the converse true? Justify your answer.

Advanced

Problem 37. A function $f(x): E \mapsto \mathbb{R}$ is called "uniformly continuous" if for any $\varepsilon > 0$, there is $\delta > 0$ such that for all $x, y \in E$ satisfying $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$.

a) Prove that if f is uniformly continuous, then it is continuous.

b) Give an example of a continuous function that is not uniformly continuous. Justify your answer.

c) If $f: E \mapsto \mathbb{R}$ is continuous with E a bounded closed set, then f is uniformly continuous.

d) Prove that if f is continuous on $[a, b]$, then it is integrable on $[a, b]$.

Problem 38. Let $f(x)$ be continuous over $\mathbb{R}$, and satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that there is $a \in \mathbb{R}$ such that $f(x) = a\, x$.

Problem 39. (USTC) Let $f(x)$ be differentiable. Assume that there are $a < b$ such that $f(a) = f(b) = 0$, $f'(a)\, f'(b) > 0$. Prove that there is $\xi \in (a, b)$ such that $f(\xi) = 0$.

Problem 40. Let $f(x)$ be continuous on (a, b). Assume there is $x_0 \in (a, b)$ such that $f'''(x_0)$ exists. Prove that there are constants A, B, C, D such that

$$\lim_{h \longrightarrow 0} \frac{A\, f(x_0 + h) + B\, f(x_0) + C\, f(x_0 - h) + D\, f(x_0 - 2\, h)}{h^3} = f'''(x_0) \tag{26}$$

and find their values. (Hint: L'Hospital)

Problem 41. (USTC) Let $f(x)$ be differentiable at x_0 with $f(x_0) \neq 0$ and $f'(x_0) = 5$. Take for granted $\lim_{h \longrightarrow 0} (1 + h)^{1/h} = e$. Calculate

$$\lim_{n \longrightarrow \infty} \left| \frac{f\left(x_0 + \frac{1}{n}\right)}{f(x_0)} \right|^n. \tag{27}$$

Problem 42. (USTC) Let f be twice differentiable over $\mathbb{R}$, with $f(0)=f(1)=0$. Let $F(x)=x^2 f(x)$. Prove that there is $\xi\in(0,1)$ such that $F''(\xi)=0$.

Problem 43. Let f be differentiable over $\mathbb{R}$. Then $f'(x)$, though may be not continuous, always satisfies the Intermediate Value Property:

For any s between $f'(a)$ and $f'(b)$, there is $\xi\in[a,b]$ such that $f'(\xi)=s$.

Then use this to prove: If f is differentiable in $(a,\ b)$ and $f'\neq 0$, then f is either increasing or decreasing.

Problem 44. (USTC) Calculate

$$\lim_{n\longrightarrow\infty}\sum_{k=1}^{n}\sin\left(\frac{k\,a}{n^2}\right). \tag{28}$$

(Hint: Write $\sum_{k=1}^{n}\sin\left(\frac{k\,a}{n^2}\right)=\sum_{k=1}^{n}\frac{k\,a}{n^2}+\sum_{k=1}^{n}\left[\sin\left(\frac{k\,a}{n^2}\right)-\frac{k\,a}{n^2}\right]$, try to estimate $\left|\sin\left(\frac{k\,a}{n^2}\right)-\frac{k\,a}{n^2}\right|$ using Taylor polynomial)

Problem 45. Let f be differentiable on $(0,\ \infty)$ with $\lim_{x\longrightarrow\infty}[f(x)+f'(x)]=0$. Prove that $\lim_{x\longrightarrow\infty}f(x)=0$. (Hint: Let $F(x)=e^x f(x)$, $G(x)=e^x$. Apply Cauchy's generalized mean value theorem.)

Problem 46. Let f be continuous on $[0,\infty)$ and satisfy $\lim_{x\longrightarrow\infty}f(x)=a$. Prove

$$\lim_{x\longrightarrow\infty}\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t=a. \tag{29}$$

Problem 47. (USTC) Let

$$F(x)=\int_0^x\frac{\sin t}{t}\,\mathrm{d}t,\qquad x\in(0,\infty). \tag{30}$$

Prove that $\max_{x\in\mathbb{R}}F=F(\pi)$.

Problem 48. $a_n\geqslant 0$, $\sum_{n=1}^{\infty}a_n$ converges. Let $b_n=\frac{a_n}{\sum_n^{\infty}a_k}$. Prove that $\sum_{n=1}^{\infty}b_n$ diverges.

Problem 49. (Alternating series) Let $b_n\geqslant 0$ with $\lim_{n\longrightarrow\infty}b_n=0$. Assume there is $N\in\mathbb{N}$ such that for all $n>N$, $b_n\geqslant b_{n+1}$.

a) Prove that $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ converges.

b) Apply this criterion to prove the convergence of $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ and $\sum_{n=1}^{\infty}(-1)^n\frac{3^n}{n!}$.

c) Show that the condition "b_n is decreasing" cannot be dropped.

Really Advanced

Problem 50. Let f be defined on (a,b) and $x_0 \in (a,b)$. Assume that $f^{(n+1)}(x)$ exists and is continuous on (a,b) with $f^{(n+1)}(x_0) \neq 0$. Consider the Taylor polynomial with Lagrange remainder:

$$f(x) = \cdots + \frac{f^{(n)}(\xi)}{n!}(x-x_0)^n. \tag{31}$$

Recall that ξ can be viewed as a function of x. If we define (naturally) $\xi(x_0) = x_0$, prove that $\xi(x)$ is differentiable at x_0 with

$$\xi'(x_0) = \frac{1}{n+1}. \tag{32}$$

Problem 51. (USTC) Let f be differentiable. $a\,b > 0$. Then there is $\xi \in (a,b)$ such that

$$\frac{1}{a-b}[a\,f(b) - b\,f(a)] = f(\xi) - \xi\,f'(\xi). \tag{33}$$

(Hint: Use Cauchy's Generalized Mean Value Theorem).

Problem 52. (USTC) Let $f(x)$ be differentiable on $[0,1]$. $f(0)=0, f(1)=1$. Then for any $n \in \mathbb{N}$ and $k_1, \ldots, k_n > 0$, there are n distinct numbers $x_1, \ldots, x_n \in (0,1)$, such that

$$\sum_{i=1}^{n} \frac{k_i}{f'(x_i)} = \sum_{i=1}^{n} k_i. \tag{34}$$

Remark 1. Note that when $k=1$, this is simply mean value theorem. Also if we do not require $x_1, \ldots,$ x_n to be distinct, the problem is trivial since we can take $x_1 = \cdots = x_n = \xi$ with $f'(\xi) = 1$.

(Hint: Take $y_1 < y_2 < \cdots < y_{n-1}$ such that $f(y_i) = \frac{k_1 + \cdots + k_i}{k_1 + \cdots + k_n}$. Set $y_0 = 0, y_1 = 1$. Then define $g(x)$ to be linear on each $[y_i, y_{i+1}]$ with $g(y_i) = f(y_i), g(y_{i+1}) = f(y_{i+1})$. Apply Cauchy's generalized mean value theorem.)

Problem 53. (USTC) Let f, g be continuous on $[-1,1]$, infinitely differentiable on $(-1,1)$, and

$$\left|f^{(n)}(x) - g^{(n)}(x)\right| \leqslant n!\,|x| \qquad n = 0, 1, 2, \ldots \tag{35}$$

Prove that $f = g$. (Hint: Show first $f^{(n)}(0) = 0$ for all n. Then use Taylor polynomial with Lagrange form of remainder)

Problem 54. Define γ_n through $\sum_{k=1}^{n-1} \frac{1}{k} = \ln n + \gamma_n$

a) Show that $\gamma_n \geqslant 0$, γ_n is increasing with respect to n.

b) Show that $\gamma_n \longrightarrow \gamma \in \mathbb{R}$.

c) Show that $\sum_1^\infty (-1)^{n+1}/n = \ln 2$.

Problem 55. (Bonar2006)

a) Let $\sum_{n=1}^\infty a_n$ be any convergent non-negative series, then there is another convergent non-negative series $\sum_{n=1}^\infty A_n$ satisfying $\lim_{n \longrightarrow \infty}(A_n/a_n) = \infty$; (Hint: Set $A_n = \frac{a_n}{\sqrt{a_n + a_{n+1} + \cdots}}$)

b) Let $\sum_{n=1}^\infty D_n$ be any divergent non-negative series, then there is another divergent non-negative series $\sum_{n=1}^\infty d_n$ satisfying $\lim_{n \longrightarrow \infty}(d_n/D_n) = 0$. (Hint: Set $d_n = D_n/(D_1 + \cdots + D_{n-1})$)

REALLY REALLY ADVANCED

Problem 56. (USTC) Let $f(x)$ be continuous on $[0,\infty)$ and be bounded. Then for every $\lambda \in \mathbb{R}$, there is $x_n \longrightarrow \infty$ such that

$$\lim_{n \longrightarrow \infty} [f(x_n+\lambda) - f(x_n)] = 0. \tag{36}$$

Problem 57. Let $f(x)$ be differentiable with $f(x_0)=0$. Further assume $|f'(x)| \leqslant |f(x)|$ for all $x > x_0$. Prove that $f(x)=0$ for all $x \geqslant x_0$.

Problem 58. (Darboux's Theorem) [1]Let $f(x)$ be a bounded function over a finite interval $[a,b]$. Let $P_n = \left\{ x_0 = a, x_1 = a + \frac{b-a}{n}, \ldots, x_n = b \right\}$. Then

$$U(f, P_n) \longrightarrow U(f); \qquad L(f, P_n) \longrightarrow L(f). \tag{37}$$

Problem 59. (Claesson1970) Let $f(x)$ be a bounded function over a finite interval $[a,b]$. Let $U(f)$ denote its upper integral. Prove: f is integrable $\Longleftrightarrow$ For any bounded function $g(x)$,

$$U(f+g) = U(f) + U(g). \tag{38}$$

1. Darboux Theorem actually states that the conclusion holds for any sequence of partitions with $\sup_i (x_i - x_{i-1}) \longrightarrow 0$. But the proof in such general case is very similar to the special one here.