6. Infinite Series

6.0. Motivating examples.

Example 6.1. (Zeno's tortoise and Achilles) The story is easy to find (say seach wiki) so omitted.

Example 6.2. (Jack and the beanstalk) Let's say the beanstalk is 10 meters long originally, and Jack can climb 1 meter every second. But at the end of each second, the beanstalk grows an extra 5 meters (assume that this takes no time, or assume Jack stop to take a breath). The question now is, will Jack be able to reach the top?

Example 6.3. (Quantum Field Theory) The following is adapted from "Quantum Field Theory" (arXiv:0204014) by R. E. Borcherds.

Life cycle of a theoretical physicist:

1. Write down a Lagrangian density L.
2. Write down the corresponding Feynman path integral.
3. Calculate Feynman path integral by expanding it into an infinite sum: $a_0 + a_1 \lambda + \cdots$
4. Work out the integrals and add everything up.
5. Realise that the integrals diverge – that is $a_0, a_1, \ldots$ are infinite.
6. Regularize the integrals, make it finite. For example instead of $\int_{-\infty}^{\infty} \frac{1}{x^2}\,\mathrm{d}x$, use $\int_{-\infty}^{-\varepsilon} \frac{\mathrm{d}x}{x^2} + \int_{\varepsilon}^{\infty} \frac{\mathrm{d}x}{x^2}$.
7. Obtain a new infinite sum where a_i's are finite.
8. Realise that the new infinite sum diverges, despite the finiteness of all the a_i's.
9. Ignore Step 8. Take only the first few terms and compare with experiment.
10. Depending on the outcome of Step 9: Collect a Nobel prize or return to Step 1.

Example 6.4. (Definition of functions) We have seen that any infinitely differentiable function corresponds to an infinite sum of monomials (it's Taylor series):

$$f(x) \Longrightarrow f(x_0) + f'(x_0)\,(x - x_0) + \frac{f''(x_0)}{2!}\,(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n. \tag{6.1}$$

It turns out that whether we can write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n \tag{6.2}$$

is a very complicated issue. For example, consider

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{6.3}$$

It can be calculated that $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$. Consequently its Taylor series reads

$$0 + 0\,(x - x_0) + 0\,(x - x_0)^2 + \cdots \tag{6.4}$$

whose only reasonable value is 0, different from $f(x)$ except at $x = 0$.

On the other hand, one can show that the Taylor series of e^x: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ indeed add up to e^x for every $x \in \mathbb{R}$.

To understand such phenomena, we need to study infinite sums of functions. But before that we need to understand infinite sum of numbers, because after all, an infinite sum of functions becomes an infinite sum of numbers once the variable x is assigned a value.

6.1. Definitions and basics.

6.1.1. Definitions.

Definition 6.5. **(Infinite series)** *Given a sequence $\{a_n\}$ of real numbers, the formal sum*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots \tag{6.5}$$

is called an "infinite series".

Remark 6.6. The "summation" of infinitely many real numbers

$$a_1 + a_2 + \cdots + a_n + \cdots \tag{6.6}$$

is "formal" because it is not clear what it means to say $s = a_1 + a_2 + \cdots + a_n + \cdots$ where $s \in \mathbb{R} \cup \{\infty, -\infty\}$.

Remark 6.7. Note that $\sum_{n=1}^{\infty} a_n$ is **just another way** of denoting the formal sum $a_1 + a_2 + \cdots + a_n + \cdots$.

Example 6.8. Some examples of infinite series:

$$\sum_{n=1}^{\infty} (-1)^n = (-1) + 1 + (-1) + 1 + \cdots \tag{6.7}$$

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\sin 1}{1} + \frac{\sin 2}{2} + \cdots \tag{6.8}$$

$$\sum_{n=1}^{\infty} 2^n = 1 + 2 + 4 + \cdots \tag{6.9}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots \tag{6.10}$$

It is clear that the value of $\sum_{n=1}^{\infty} \frac{1}{2^n}$ should be 2 while $\sum_{n=1}^{\infty} 2^n$ should be ∞. It's not clear whether the first two sums corresponds to any value.

Definition 6.9. **(Partial sum and convergence)** *The nth partial sum of an infinite series $\sum_{n=1}^{\infty} a_n$ is defined as $s_n = \sum_{m=1}^{n} a_m$. If the sequence $\{s_n\}$ converges to some real number s, then we say the infinite series converges, and say its sum is s, and simply write*

$$\sum_{n=1}^{\infty} a_n = s. \tag{6.11}$$

If $s \longrightarrow \infty$ or $-\infty$, we say the infinite series diverges to ∞ or $-\infty$ respectively and write

$$\sum_{n=1}^{\infty} a_n = \infty \text{ or } -\infty. \tag{6.12}$$

Recalling theorems for the convergence of sequences, we have

Theorem 6.10.

- $\sum_{n=1}^{\infty} a_n = s$ *if and only if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$,*

$$\left| s - \sum_{m=1}^{n} a_m \right| < \varepsilon; \tag{6.13}$$

- $\sum_{n=1}^{\infty} a_n = \infty$ *if and only if for any $M \in \mathbb{R}$, there is $N \in \mathbb{N}$ such that for all $n > N$,*

$$\sum_{m=1}^{n} a_m > M; \tag{6.14}$$

- $\sum_{n=1}^{\infty} a_n = -\infty$ *if and only if for any $M \in \mathbb{R}$, there is $N \in \mathbb{N}$ such that for all $n > N$,*

$$\sum_{m=1}^{n} a_m < M. \tag{6.15}$$

Proof. Left as exercises. □

Theorem 6.11. (Cauchy) *A infinite series* $\sum_{n=1}^{\infty} a_n$ *converges if and only if for any* $\varepsilon > 0$ *there is* $N \in \mathbb{N}$ *such that for all* $m > n > N$,

$$\left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon. \tag{6.16}$$

Proof. Left as exercise. □

Corollary 6.12. *If* $\sum_{n=1}^{\infty} a_n$ *converges to* $s \in \mathbb{R}$ *then* $\lim_{n \longrightarrow \infty} a_n = 0$. *Equivalently, if* $\lim_{n \rightarrow \infty} a_n$ *does not exist, or exists but is not* 0, *then* $\sum_{n=1}^{\infty} a_n$ *does not converge to any real number.*

Proof. For any $\varepsilon > 0$, since $\sum_{n=1}^{\infty} a_n$ converges, it is Cauchy and there exists $N_1 \in \mathbb{N}$ such that for all $m > n > N_1$,

$$\left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon. \tag{6.17}$$

Now take $N = N_1 + 1$. Then for any $n > N$, we have $n > n - 1 > N_1$ which gives

$$|a_n - 0| = \left| \sum_{k=n}^{n} a_k \right| < \varepsilon. \tag{6.18}$$

Thus by definition of convergence of sequence $\lim_{n \longrightarrow \infty} a_n = 0$. □

Remark 6.13. Corollary 6.12 is very useful, however we should keep in mind that:

1. The converse is not true. That is $\lim_{n \rightarrow \infty} a_n = 0$ does not imply the convergence of $\sum_{n=1}^{\infty} a_n$.
2. It cannot be applied to conclude $\sum_{n=1}^{\infty} a_n \neq \infty$ or $-\infty$.

Example 6.14. Let $a_n = r^{n-1}$ for $r \in \mathbb{R}$. Then

a) If $|r| < 1$, then $\sum_{n=1}^{\infty} a_n = 1 + r + r^2 + \cdots = \frac{1}{1-r}$.

b) If $r \geqslant 1$, then $\sum_{n=1}^{\infty} a_n = \infty$.

c) If $r \leqslant -1$, then $\sum_{n=1}^{\infty} a_n$ does not exist (as extended real number).

Proof.

a) We have

$$\sum_{m=1}^{n} a_m = 1 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}. \tag{6.19}$$

For any $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $N \geqslant \log_{|r|} [\varepsilon (1 - r)]$, then for any $n > N$,

$$\left| \frac{1}{1-r} - \sum_{m=1}^{n} a_m \right| = \frac{|r|^n}{1-r} < \frac{|r|^N}{1-r} < \varepsilon. \tag{6.20}$$

b) For any $M \in \mathbb{R}$. Take $N \in \mathbb{N}$ such that $N > |M|$. Then for every $n > N$ we have

$$\sum_{m=1}^{n} a_m \geqslant \sum_{m=1}^{n} 1 = n > N > |M| \geqslant M. \tag{6.21}$$

c) Since $r \leqslant -1$, $|a_n| \geqslant 1$. Therefore by Corollary 6.12 $\sum_{n=1}^{\infty} a_n$ does not converge to any real number. We still need to show that $\sum_{n=1}^{\infty} a_n \neq \infty, -\infty$. To do this, we show that $s_n = \sum_{m=1}^{n} a_n$ satisfies $s_n \geqslant 0$ when n is odd and $s_n \leqslant 0$ when n is even. Clearly $s_1 = 1 > 0$, $s_2 = 1 + r \leqslant 0$. For $n \geqslant 3$, calculate

$$s_n = \sum_{m=1}^{n-1} r^{m-1} + r^n = \frac{1 - r^{n-1}}{1 - r} + r^n. \tag{6.22}$$

As $r \leqslant -1$, $1 - r \geqslant 2$ which gives

$$\left| \frac{1 - r^{n-1}}{1 - r} \right| \leqslant \frac{1 + |r|^{n-1}}{2} \leqslant |r|^{n-1} \leqslant |r|^n. \tag{6.23}$$

Therefore, s_n and r^n cannot take opposite signs. □

Example 6.15. We have

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{1 - \frac{1}{2}} = 2; \qquad \sum_{n=1}^{\infty} \frac{1}{5^{n-1}} = \frac{5}{4}. \tag{6.24}$$

Example 6.16. Consider

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\sin 1}{1} + \frac{\sin 2}{2} + \cdots \tag{6.25}$$

We show that it actually converges.

In light of Theorem 6.11, we need to understand

$$\sum_{k=n+1}^{m} \frac{\sin k}{k}. \tag{6.26}$$

Denote $A_k = \sum_{l=1}^{k} \sin l$, and $B_k = \frac{1}{k}$. Then we have

$$\begin{aligned}
\sum_{k=n+1}^{m} \frac{\sin k}{k} &= \sum_{k=n+1}^{m} (A_k - A_{k-1}) B_k \\
&= [A_{n+1} B_{n+1} - A_n B_{n+1}] + [A_{n+2} B_{n+2} - A_{n+1} B_{n+2}] + \cdots \\
&\quad + [A_m B_m - A_{m-1} B_m] \\
&= [A_m B_m - A_n B_{n+1}] + \\
&\quad + [A_{n+1} (B_{n+1} - B_{n+2}) + A_{n+2} (B_{n+2} - B_{n+3}) + \cdots + A_{m-1} (B_{m-1} - B_m)] \\
&= [A_m B_m - A_n B_{n+1}] + \sum_{k=n+1}^{m-1} [A_k (B_k - B_{k+1})] \\
&= \left[\frac{A_m}{m} - \frac{A_n}{n+1} \right] + \sum_{k=n+1}^{m-1} \left[A_k \left(\frac{1}{k} - \frac{1}{k+1} \right) \right].
\end{aligned} \tag{6.27}$$

Now notice that $\{A_n\}$ is in fact a bounded sequence:

$$\begin{aligned}
A_n &= \sin 1 + \sin 2 + \cdots + \sin n \\
&= \frac{\sin 1 \, [\sin 1 + \sin 2 + \cdots + \sin n]}{\sin 1} \\
&= \frac{\cos (1-1) - \cos (1+1) + \cos (2-1) - \cos (2+1) + \cdots + \cos (n-1) - \cos (n+1)}{2 \sin 1} \\
&= \frac{[\cos 0 + \cos 1 + \cdots + \cos (n-1)] - [\cos 2 + \cos 3 + \cdots + \cos (n+1)]}{2 \sin 1} \\
&= \frac{\cos 0 + \cos 1 - \cos n - \cos (n+1)}{2 \sin 1}.
\end{aligned} \tag{6.28}$$

Now it is clear that $|A_n| \leqslant \frac{2}{\sin 1}$ for all $n \in \mathbb{N}$.

Back to (6.27):

$$\begin{aligned}
\left| \sum_{k=n+1}^{m} \frac{\sin k}{k} \right| &\leqslant \left| \frac{A_m}{m} \right| + \left| \frac{A_n}{n+1} \right| + \sum_{k=n+1}^{m-1} |A_k| \left(\frac{1}{k} - \frac{1}{k+1} \right) \\
&\leqslant \frac{2}{\sin 1} \left[\frac{1}{m} + \frac{1}{n+1} + \sum_{k=n+1}^{m-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) \right] \\
&= \frac{2}{\sin 1} \left[\frac{2}{n+1} \right] = \frac{4}{(n+1) \sin 1}.
\end{aligned} \tag{6.29}$$

Now we are ready to show the series is Cauchy:

For any $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $N + 1 \geqslant \frac{4}{\varepsilon \sin 1}$, then for any $m > n > N$, we have

$$\left| \sum_{k=n+1}^{m} \frac{\sin k}{k} \right| \leqslant \frac{4}{(n+1)\sin 1} < \frac{4}{(N+1)\sin 1} \leqslant \varepsilon. \tag{6.30}$$

Therefore the series converges.

6.1.2. Operations of infinite series.

Theorem 6.17. (Arithmetics) *If $\sum_{n=1}^{\infty} a_n = s, \sum_{n=1}^{\infty} b_n = t$ with s, t extended numbers. Then*

- *For any $c \in \mathbb{R}$, if $c\, s$ is defined, $\sum_{n=1}^{\infty} (c\, a_n) = c\, s$.*
- *If $s + t$ is defined, $\sum_{n=1}^{\infty} (a_n + b_n) = s + t$.*

Remark 6.18. Note that when $c = 0$ and $s = \infty$ or $-\infty$, although the product $c\, s$ is not defined, we can easily prove using definition that $\sum_{n=1}^{\infty} (c\, a_n) = \sum_{n=1}^{\infty} 0 = 0$.

Example 6.19. Let $a_n = c\, r^{n-1}$ for $r \in \mathbb{R}$ and $c \in \mathbb{R}$. Then

a) If $|r| < 1$, then $\sum_{n=1}^{\infty} a_n = 1 + r + r^2 + \cdots = \frac{c}{1-r}$.

b) If $r \geqslant 1$ and $c \neq 0$, then $\sum_{n=1}^{\infty} a_n = c \cdot \infty = \begin{cases} \infty & c > 0 \\ -\infty & c < 0 \end{cases}$.

c) If $r \leqslant -1$ and $c \neq 0$, then $\sum_{n=1}^{\infty} a_n$ does not exist (as extended real number).

d) If $c = 0$ then $\sum_{n=1}^{\infty} a_n = 0$ no matter what value r takes.

a),b),d) clearly follow from Theorem 6.17. We prove c) by contradiction: If $\sum_{n=1}^{\infty} a_n = s \in \mathbb{R} \cup \{-\infty, \infty\}$, then since $c^{-1} \neq 0$, we have

$$\sum_{n=1}^{\infty} r^{n-1} = \sum_{n=1}^{\infty} (c^{-1} a_n) = c^{-1} s \in \mathbb{R} \cup \{-\infty, \infty\}. \tag{6.31}$$

Contradiction.

Example 6.20. We have

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{5^{n-1}} = \frac{1}{5} \frac{5}{4} = \frac{1}{4}. \tag{6.32}$$

Remark 6.21. In general there is no relation between $\sum_{n=1}^{\infty} a_n b_n$ and $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$. On the other hand, with some extra assumption we can define the product

$$\left(\sum_{n=1}^{\infty} a_n \right) \left(\sum_{n=1}^{\infty} b_n \right) = \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} a_k\, b_{n+1-k} \right] = (a_1 b_1) + (a_1 b_2 + a_2 b_1) + (a_1 b_3 + a_2 b_2 + a_3 b_1) + \cdots \tag{6.33}$$

This will be discussed in Math414.

6.1.3. Forbidden operations.

Example 6.22. (Grouping) Unless $a_n \geqslant 0$ (or all $\leqslant 0$) and $\sum_{n=1}^{\infty} a_n$ converges, the order of summation cannot be changed. For example let $a_n = (-1)^{n+1}$. If we are allowed to group terms together and sum them first, we would have both

$$\sum_{n=1}^{\infty} a_n = 1 + (-1) + 1 + \cdots = 1 + [(-1) + 1] + [(-1) + 1] + \cdots = 1 + 0 + 0 + \cdots = 1; \tag{6.34}$$

$$\sum_{n=1}^{\infty} a_n = 1 + (-1) + 1 + \cdots = [1 + (-1)] + [1 + (-1)] + \cdots = 0 + 0 + 0 + \cdots = 0. \tag{6.35}$$

Definition 6.23. (Rearrangement) *A rearrangement of an infinite series* $\sum_{n=1}^{\infty} a_n$ *is another infinite series* $\sum_{m=1}^{\infty} a_{n(m)}$ *where* $m: \mapsto n(m)$ *is a bijection from* $\mathbb{N}$ *to* $\mathbb{N}$.

Example 6.24. An example of rearrangement of $a_1 + a_2 + a_3 + \cdots$ is $a_2 + a_4 + a_7 + a_1 + a_5 + a_3 + a_6 + \cdots$.

Example 6.25. (Rearrangement) Consider the sequence $\sum_{n=1}^{\infty} a_n$ with $a_n = \frac{(-1)^{n+1}}{n}$. If we are allowed to freely rearrange (that is choose the order of summation), then for any $s \in \mathbb{R} \cup \{-\infty, \infty\}$, there is a rearrangement such that it converges to s.

Proof. Consider the case $s \in \mathbb{R}$. The cases $s = \infty, -\infty$ are left as exercises.

Consider the rearrangement $\sum_{n=1}^{\infty} b_n$ defined as follows:

- Let k_0 be such that $1 + \frac{1}{3} + \cdots + \frac{1}{2\,k_0 - 1} \geqslant s$ but $1 + \frac{1}{3} + \cdots + \frac{1}{2\,k_0 - 3} < s$. Set

$$b_1 = 1, b_2 = \frac{1}{3}, \ldots, b_{k_0} = \frac{1}{2\,k_0 - 1}; \tag{6.36}$$

  The case $k_0 = 1$ is when $1 \geqslant s$. Then we just set $b_1 = 1$ and turn to the next step.

- Let k_1 be such that

$$\sum_{k=1}^{k_0} b_k - \left(\frac{1}{2} + \cdots + \frac{1}{2\,k_1 - 2}\right) \geqslant s, \qquad \sum_{k=1}^{k_0} b_k - \left(\frac{1}{2} + \cdots + \frac{1}{2\,k_1}\right) < s \tag{6.37}$$

  and set

$$b_{k_0+1} = -\frac{1}{2}, \qquad b_{k_0+k_1} = -\frac{1}{2\,(k_1+1)}. \tag{6.38}$$

- Let k_2 be such that

$$\sum_{k=1}^{k_0+k_1} b_k + \left(\frac{1}{2\,k_0+1} + \cdots + \frac{1}{2\,k_0 + 2\,k_2 - 1}\right) \geqslant s, \sum_{k=1}^{k_0+k_1} b_k + \left(\frac{1}{2\,k_0+1} + \cdots + \frac{1}{2\,k_0 + 2\,k_2 - 3}\right) < s \tag{6.39}$$

  and set

$$b_{k_0+k_1+1} = \frac{1}{2\,k_0+1}, \ldots, b_{k_0+k_1+k_2} = \frac{1}{2\,k_0 + 2\,k_2 + 1}, \tag{6.40}$$

- And so on.

Now set

$$S_l = \sum_{k=1}^{k_0+k_1+\cdots+k_l} b_k. \tag{6.41}$$

Then we see that if $n \in [k_0 + \cdots + k_l, k_0 + \cdots + k_{l+1}]$, then

$$s_n = \sum_{m=1}^{n} b_m \tag{6.42}$$

is always between S_l and S_{l+1}.

Finally notice that by construction, $|S_l - s| < \frac{1}{l}$. Thus for any $\varepsilon > 0$, take $L \in \mathbb{N}$ such that $L > \varepsilon^{-1}$. Now set $N = k_0 + \cdots + k_L$. For any $n > N$, there is $l \geqslant L$ such that $n \in [k_0 + \cdots + k_l, k_0 + \cdots + k_{l+1}]$. Therefore we have

$$|s_n - s| \leqslant \max\{|S_l - s|, |S_{l+1} - s|\} \leqslant \frac{1}{l} \leqslant \frac{1}{L} < \varepsilon. \tag{6.43}$$

That is $\sum_{n=1}^{\infty} b_n \longrightarrow s$ by definition. □

Remark 6.26. Note that the above proof depends on the fact that

$$\sum_{k=1}^{\infty} \frac{1}{2\,k-1} = \infty, \qquad \sum_{k=1}^{\infty} \left(-\frac{1}{2\,k}\right) = -\infty. \tag{6.44}$$

These two facts are left as exercises.

6.2. Non-negative series.

An infinite series $\sum_{n=1}^{\infty} a_n$ is called "non-negative" if $a_n \geqslant 0$ for all $n \in \mathbb{N}$.

6.2.1. The importance of non-negative series.

Theorem 6.27. *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two infinite series. Assume that there are $c > 0$ and $N_0 \in \mathbb{N}$ such that $|a_n| \leqslant c\, b_n$ for all $n > N_0$. Then*

a) $\sum_{n=1}^{\infty} b_n$ converges $\Longrightarrow$ $\sum_{n=1}^{\infty} a_n$ converges.

b) $\sum_{n=1}^{\infty} a_n$ diverges $\Longrightarrow$ $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. Note that a) and b) are equivalent logical statements, so we only need to prove a). We show that $\sum_{n=1}^{\infty} a_n$ is Cauchy. For any $\varepsilon > 0$, since $\sum_{n=1}^{\infty} b_n$ converges, it is Cauchy and there is $N_1 \in \mathbb{N}$ such that for all $m > n > N_1$,

$$\left| \sum_{k=n+1}^{m} b_k \right| < \frac{\varepsilon}{c}. \tag{6.45}$$

Take $N = \max\{N_1, N_0\}$. Then for any $m > n > N$,

$$\left| \sum_{k=n+1}^{m} a_k \right| \leqslant \sum_{k=n+1}^{m} |a_k| \leqslant c \left| \sum_{k=n+1}^{m} b_k \right| < \varepsilon. \tag{6.46}$$

So $\sum_{n=1}^{\infty} a_n$ is Cauchy and therefore converges. □

Example 6.28. It is clear by Theorem 6.27 that if $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} a_n$. The converse is not true, as can be seen from the following example:

Take $a_n = \frac{(-1)^{n+1}}{n}$. Then we clearly see that

$$S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(2n-1)(2n)} \tag{6.47}$$

converges. On the other hand, we have $S_{2n+1} - S_{2n} \longrightarrow 0$ so S_{2n+1} converges to the same limit. From here it is easy to prove by definition that $S_n \longrightarrow$ to the same limit, which turns out to be $\ln 2$.

Remark 6.29. A sequence $\sum_{n=1}^{\infty} a_n$ that converges but with $\sum_{n=1}^{\infty} |a_n| = \infty$ is called conditionally convergent. It turns out that the phenomenon we have seen in Example 6.25 is quite generic for conditionally convergent series. More specifically, if $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then it can be re-arranged to converge to any extended real number.

On the other hand, a sequence $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} |a_n|$ converges is said to be *absolutely convergent*. Absolutely convergent sequences can undergo any re-arrangement and still converge to the same sum.

6.2.2. Typical non-negative series and their implications.

In light of Theorem 6.27, it is important to understand the convergence/divergence of non-negative infinite series. Once a non-negative sequence $\sum_{n=1}^{\infty} b_n$ is shown to be convergent, we know that any $\sum_{n=1}^{\infty} a_n$ satisfying $|a_n| \leqslant c\ b_n$ for some constant c is also convergent. It is further possible to make this comparison "intrinsic", that is design some criterion involving a_n only and guarantees the relation $|a_n| \leqslant c\, b_n$. Such criteria are usually called "tests". We will study some simple tests now.

First we note that non-negative series converges if and only if it is bounded above.

Theorem 6.30. *Let $\sum_{n=1}^{\infty} a_n$ be a non-negative series. Then it converges $\Longleftrightarrow$ it is bounded above.*

Proof. Note that the partial sum $s_n = \sum_{n=1}^{\infty} a_n$ is increasing. The rest is left as exercise. □

Example 6.31. (Geometric series) We have seen that $\sum_{n=1}^{\infty} r^{n-1}$ converges when $0 \leqslant r < 1$. As a consequence, if another series $\sum_{n=1}^{\infty} a_n$ satisfies

$$|a_n| \leqslant c\, r^{n-1} \tag{6.48}$$

for some $c>0$ and for all $n>$ some $N_0\in\mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ converges.

The following two intrinsic "convergence tests" based on comparison with geometric series are the simplest and most popular tests for convergence/divergence.

Theorem 6.32. (Ratio test) *Let $\sum_{n=1}^{\infty} a_n$ a infinite series. Further assume that $a_n\neq 0$ for all $n\in\mathbb{N}$ (see Remark 6.34 though). Then*

- *If* $\limsup_{n\longrightarrow\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$, *the series converges.*
- *If* $\liminf_{n\longrightarrow\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$, *the series diverges.*

Proof.

- Assume $\limsup_{n\longrightarrow\infty}\left|\frac{a_{n+1}}{a_n}\right|=L<1$. Set $r=\frac{L+1}{2}$ and $\varepsilon_0=\frac{1-L}{2}$. By definition

$$\limsup_{n\longrightarrow\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\longrightarrow\infty}\left\{\sup_{k\geqslant n}\left|\frac{a_{k+1}}{a_k}\right|\right\} \tag{6.49}$$

therefore there is $N\in\mathbb{N}$ such that for all $n>N$,

$$\left|\sup_{k\geqslant n}\left|\frac{a_{k+1}}{a_k}\right|-L\right|<\varepsilon_0 \tag{6.50}$$

which gives

$$\left|\sup_{n>N}\left|\frac{a_{n+1}}{a_n}\right|\right|<L+\varepsilon_0=r<1\Longrightarrow 0<\left|\frac{a_{n+1}}{a_n}\right|<r<1. \tag{6.51}$$

This gives, for all $n>N+1$,

$$|a_n|<|a_{N+1}|\,r^{n-N-1}=\frac{|a_{N+1}|}{r^N}\,r^{n-1}. \tag{6.52}$$

Note that since N is fixed, we have

$$|a_n|<c\,r^{n-1} \tag{6.53}$$

for all $n>N$ and consequently $\sum_{n=1}^{\infty} a_n$ converges.

- Assume $\liminf_{n\longrightarrow\infty}\left|\frac{a_{n+1}}{a_n}\right|=L>1$. Set $\varepsilon_0=L-1$. Then by definition, similar to the limsup case above, there is $N\in\mathbb{N}$ such that for all $n>N$,

$$\left|\frac{a_{n+1}}{a_n}\right|>1 \tag{6.54}$$

which means for all $n\geqslant N+1$

$$|a_n|\geqslant|a_{N+1}| \tag{6.55}$$

As a consequence $a_n\not\longrightarrow 0$. By Corollary 6.12 we know that $\sum_{n=1}^{\infty} a_n$ diverges. □

Example 6.33. Prove that $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n!}}$ converges.

Proof. We have

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{\sqrt{n+1}} \tag{6.56}$$

therefore

$$\limsup_{n\longrightarrow\infty}\left|\frac{a_{n+1}}{a_n}\right|=0<1. \tag{6.57}$$

So the series converges. □

Remark 6.34. Theorem 6.32 is applicable to non-negative infinite series – we just need to first drop all the zero terms!

Theorem 6.35. (Root test) *Let* $\sum_{n=1}^{\infty} a_n$ *be a infinite series. Then*

- *If* $\limsup_{n\longrightarrow\infty} |a_n|^{1/n} < 1$*, then the series converges.*
- *If* $\liminf_{n\longrightarrow\infty} |a_n|^{1/n} > 1$*, then the series diverges.*

Proof.

- Assume $\limsup_{n\longrightarrow\infty} |a_n|^{1/n} = L < 1$. Set $r = \frac{L+1}{2}$ and $\varepsilon_0 = \frac{1-L}{2}$. Then by definition, as in the proof of the above ratio test, there is $N \in \mathbb{N}$ such that for all $n > N$,
$$|a_n|^{1/n} < r < 1 \Longrightarrow |a_n| < r^n. \tag{6.58}$$
Therefore $\sum_{n=1}^{\infty} a_n$ converges.
- Assume $\liminf_{n\longrightarrow\infty} |a_n|^{1/n} > 1$. The proof is left as exercise. □

Remark 6.36. Recall Problem 29 of Midterm Practice, for $x_n > 0$
$$\liminf_{n\longrightarrow\infty} \frac{x_{n+1}}{x_n} \leqslant \liminf_{n\longrightarrow\infty} (x_n)^{1/n} \leqslant \limsup_{n\longrightarrow\infty} (x_n)^{1/n} \leqslant \limsup_{n\longrightarrow\infty} \frac{x_{n+1}}{x_n}. \tag{6.59}$$
Therefore the root test is sharper than the ratio test, in the sense that any series that passes the ratio test for convergence will also pass the root test.

Example 6.37. Consider the infinite series $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} a_n$ where $a_n = \begin{cases} 2^{-k} & n = 2k-1 \\ 3^{-k} & n = 2k \end{cases}$. It can be easily verified that
$$\liminf_{n\longrightarrow\infty} \frac{a_{n+1}}{a_n} = 0; \qquad \limsup_{n\longrightarrow\infty} \frac{a_{n+1}}{a_n} = \infty \tag{6.60}$$
so the ratio test does not apply. On the other hand
$$\limsup_{n\longrightarrow\infty} (a_n)^{1/n} = \frac{1}{\sqrt{2}} \tag{6.61}$$
so the root test tells us that the series converges.

Example 6.38. (Generalized harmonic series) The series $\sum_{n=1}^{\infty} \frac{1}{n}$, which we know diverges, is called the harmonic series, allegedly from the length strings in a harp. We can generalize it to $\sum_{n=1}^{\infty} \frac{1}{n^a}$ for $a > 0$. It turns out that this series converges when $a > 1$ and diverges when $a \leqslant 1$.

Proof. When $a \leqslant 1$, we have $\frac{1}{n^a} \geqslant \frac{1}{n}$ therefore the divergence of $\sum_{n=1}^{\infty} \frac{1}{n}$ gives the divergence of $\sum_{n=1}^{\infty} \frac{1}{n^a}$ for $0 < a < 1$.

When $a > 1$, we use the following trick:
$$\frac{1}{n^a} = \int_{n-1}^{n} \frac{1}{n^a} \mathrm{d}x \leqslant \int_{n-1}^{n} \frac{\mathrm{d}x}{x^a}. \tag{6.62}$$
Thus
$$\sum_{n=1}^{\infty} \frac{1}{n^a} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^a} \leqslant 1 + \sum_{n=2}^{\infty} \int_{n-1}^{n} \frac{\mathrm{d}x}{x^a} = 1 + \int_{1}^{\infty} \frac{\mathrm{d}x}{x^a} = 1 + \frac{1}{a-1} = \frac{a}{a-1}. \tag{6.63}$$
We see that the series is bounded above and therefore converges. □

Remark 6.39. Recall that weeks ago we proved the convergence of the $a = 2$ case through the trick $\frac{1}{n(n-1)} \geqslant \frac{1}{n^2}$, however for more exotic a such tricks are not available anymore.

Remark 6.40. Also note that neither the ratio test nor the root test works for the generalized harmonic series.

Remark 6.41. It is also possible to design convergence/divergence tests using generalized harmonic series as the gauge. Since $\frac{1}{n^a}$ converges to 0 slower than r^n (in the sense that $\lim_{n\longrightarrow\infty} n^a r^n = 0$ if $|r| < 1$), these tests will be more refined than either the ratio test or the root test. One of such test is the following

(Raabe's test) $a_n > 0$. Then

- $\sum_{n=1}^{\infty} a_n$ converges if
$$\liminf_{n\longrightarrow\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1 \tag{6.64}$$
- $\sum_{n=1}^{\infty} a_n$ diverges if
$$\limsup_{n\longrightarrow\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1. \tag{6.65}$$

Note that the "convergent" part of Raabe's test can be turned into a convergence test for general series as

$$\liminf_{n\longrightarrow\infty} n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) > 1 \Longrightarrow \sum_{n=1}^{\infty} a_n \text{ converges} \tag{6.66}$$

but the divergent part cannot, that is

$$\limsup_{n\longrightarrow\infty} n\left(\frac{|a_n|}{|a_{n+1}|} - 1\right) < 1 \Longrightarrow \sum_{n=1}^{\infty} a_n \text{ diverges} \tag{6.67}$$

is not true due to the existence of convergent sequences such as $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

If we consider even slower convergent series, such as

$$\sum_{n=1}^{\infty} \frac{1}{n\,(\log n)^{\alpha}}, \qquad \sum_{n=1}^{\infty} \frac{1}{n\,(\log n)\,(\log(\log n))\cdots(\log(\cdots\log n))^{\alpha}} \tag{6.68}$$

for $\alpha > 1$ (the proof of convergence of these series is similar to that of the generalized harmonic series), we will obtain even sharper tests (Gauss' test from the former, Bertrand's test from the latter), but the formulas become quite baroque and few can remember them.

Finally let's look at a fun example.

Example 6.42. We know that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞. Now consider the sequence obtained by dropping all terms involving 9:

$$1 + \frac{1}{2} + \cdots + \frac{1}{8} + \cancel{\frac{1}{9}} + \frac{1}{10} + \cdots + \frac{1}{18} + \cancel{\frac{1}{19}} + \cdots + \frac{1}{88} + \cancel{\frac{1}{89}} + \cancel{\frac{1}{90}} + \cancel{\frac{1}{91}} + \cancel{\frac{1}{92}} + \cdots \tag{6.69}$$

Does this series converge?

As it is a non-negative sequence, we only need to check whether it's bounded above (Theorem 6.30). We have,

$$1 + \cdots + \frac{1}{8} < 8, \tag{6.70}$$

$$\left(\frac{1}{10} + \cdots + \frac{1}{18}\right) + \cdots + \left(\frac{1}{80} + \cdots \frac{1}{88}\right) < 8 \cdot \frac{9}{10} \tag{6.71}$$

In general, there are $8 \cdot 9^k$ terms between 10^{-k} and $\frac{1}{10^{k+1}-1}$, so their sum is bounded above by $8 \cdot \left(\frac{9}{10}\right)^k$. Overall the sum is bounded above by

$$\sum_{n=0}^{\infty} 8 \cdot \left(\frac{9}{10}\right)^n = 8 \cdot \frac{1}{1 - \frac{9}{10}} = 80. \tag{6.72}$$

Therefore the new series converges.

Remark 6.43. Obviously we can try to study the sequence resulted from deleting all terms involving other digits, or sequence of numbers. For example we can delete all terms involving the combination 43, that is $\frac{1}{4352}$ is deleted while $\frac{1}{4537}$ is not. We can even play some silly games such as deleting all terms involving 121221, or someone's birthday. The resulting sequences are all convergent.

Example 6.44. (Fermat's Last Theorem) This is taken from the blog of Terence Tao of UCLA[6.1]. We all know that Fermat's Last Theorem claims that

$$x^n + y^n = z^n, \qquad x, y, z \in \mathbb{N} \tag{6.73}$$

does not have any solution when $n \geqslant 3$. On the other hand, it is well-known that when $n = 2$, there are infinitely many solutions. But why? What's the difference between $n = 2$ and $n > 2$? It turns out that we can realize some difference through knowledge of convergence/divergence of infinite series.

Let's consider the chance of three numbers $a, b, a + b$ are all the nth power of a natural number. If we treat a as a typical number of size a, then it's chance of being an nth power is roughly $a^{1/n}/a$. Ignoring the relation between $a, b, a + b$, we have the following probability for $a, b, a + b$ solving the equation (6.73):

$$a^{\frac{1}{n}-1}\, b^{\frac{1}{n}-1}\, (a+b)^{\frac{1}{n}-1}. \tag{6.74}$$

Now consider all numbers a, b, we sum up the probabilities:

$$I := \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \left[a^{\frac{1}{n}-1}\, b^{\frac{1}{n}-1}\, (a+b)^{\frac{1}{n}-1} \right] \tag{6.75}$$

and apply the following intuition based on the so-called Borel-Cantelli Lemma in probability:

> If $I < \infty$, then the chance of (6.73) having a solution is very low, while if $I = \infty$, the chance is very high.

We notice that I has perfect symmetry between a and b, which means

$$I = 2 \sum_{a=1}^{\infty} \sum_{b=1}^{a-1} \left[a^{\frac{1}{n}-1}\, b^{\frac{1}{n}-1}\, (a+b)^{\frac{1}{n}-1} \right] + \sum_{a=1}^{\infty} a^{\frac{2}{n}-2}\, (2\,a)^{\frac{1}{n}-1}. \tag{6.76}$$

It is clear that the second series converges for all $n \geqslant 2$ so can be ignored for our purpose.

Now consider

$$J = \sum_{a=1}^{\infty} \sum_{b=1}^{a-1} \left[a^{\frac{1}{n}-1}\, b^{\frac{1}{n}-1}\, (a+b)^{\frac{1}{n}-1} \right]. \tag{6.77}$$

When $1 \leqslant b \leqslant a - 1$, we have $a < a + b < 2\,a$ therefore

$$\sum_{a=1}^{\infty} \sum_{b=1}^{a-1} a^{\frac{1}{n}-1}\, b^{\frac{1}{n}-1}\, (2\,a)^{\frac{1}{n}-1} < J < \sum_{a=1}^{\infty} \sum_{b=1}^{a-1} a^{\frac{1}{n}-1}\, b^{\frac{1}{n}-1}\, (a)^{\frac{1}{n}-1} \tag{6.78}$$

which gives

$$2^{\frac{1}{n}-1} \sum_{a=1}^{\infty} \sum_{b=1}^{a-1} a^{\frac{2}{n}-2}\, b^{\frac{1}{n}-1} < J < \sum_{a=1}^{\infty} \sum_{b=1}^{a-1} a^{\frac{2}{n}-2}\, b^{\frac{1}{n}-1}. \tag{6.79}$$

So finally all we need to study is the convergence/divergence of

$$K = \sum_{a=1}^{\infty} \sum_{b=1}^{a-1} a^{\frac{2}{n}-2}\, b^{\frac{1}{n}-1} = \sum_{a=1}^{\infty} \left[a^{\frac{2}{n}-2} \sum_{b=1}^{a-1} b^{\frac{1}{n}-1} \right]. \tag{6.80}$$

We have (see HW7 Prob. 6)

$$\sum_{b=1}^{a-1} b^{\frac{1}{n}-1} \sim \int_1^{a-1} x^{\frac{1}{n}-1}\, \mathrm{d}x \sim a^{\frac{1}{n}}. \tag{6.81}$$

Therefore

$$K \sim \sum_{a=1}^{\infty} a^{\frac{3}{n}-2} \tag{6.82}$$

which is convergent when $n \geqslant 4$ while divergent when $n = 2$.

6.1. The probabilistic heuristic justification of the ABC conjecture, link at http://terrytao.wordpress.com/2012/09/18/the-probabilistic-heuristic-justification-of-the-abc-conjecture/

Remark 6.45. The case $n = 3$ is a bit tricky here. The series $\sum_{a=1}^{\infty} a^{\frac{3}{n}-2}$ becomes the Harmonic series $\sum_{a=1}^{\infty} a^{-1}$ which is the borderline between convergence and divergence. Our argument does not provide any insight on why $x^3 + y^3 = z^3$ should not have solutions.