MATH 314 A1 FALL 2012 HOMEWORK 5

DUE FRIDAY NOV. 9.

5:30pm (Assignment box CAB 3rd floor)

- Related sections in the notes: §4.1 – §4.3.

Problem 1. (4 pts)

a) **(2 pts)** Prove **by definition** that $f(x) = |x|$ is not differentiable at $x_0 = 0$ but differentiable at all $x_0 \neq 0$. Find $f'(x)$ for $x \neq 0$.

b) **(1 pt)** Prove that if f is differentiable at x_0, then

$$\lim_{h \longrightarrow 0} \frac{f(x_0 + h) - f(x_0 - h)}{2\,h} = f'(x_0). \tag{1}$$

c) **(1 pt)** Is it true that if

$$\lim_{h \longrightarrow 0} \frac{f(x_0 + h) - f(x_0 - h)}{2\,h} = L \in \mathbb{R} \tag{2}$$

then f is differentiable at x_0? Justify your answer.

Problem 2. (3 pts) Calculate derivatives of the following functions. You can use $(e^x)' = e^x$, $(\cos x)' = -\sin x$, $(\sin x)' = \cos x$, $(\ln x)' = \frac{1}{x}$.

$$f_1(x) = \cos\left[\frac{e^x\,(x-2)}{\sin x}\right] \qquad x \neq 0; \qquad f_2(x) = \ln|x| \qquad x \neq 0; \qquad f_3(x) = \arccos x \tag{3}$$

Problem 3. (4 pts)

a) **(2 pt)** Apply L'Hospital's rule to prove $\lim_{t \longrightarrow \infty} t^n\,e^{-t} = 0$ for all $n \in \mathbb{N}$.

b) **(2 pts)** Prove that $f(x) = \begin{cases} \exp\left[-\frac{1}{x}\right] & x > 0 \\ 0 & x \leqslant 0 \end{cases}$ is differentiable at all $x \in \mathbb{R}$. Find $f'(x)$.

Problem 4. (3 pts) Prove

$$3 \arccos x - \arccos(3\,x - 4\,x^3) = \pi, \qquad |x| < 1/2. \tag{4}$$

Problem 5. (3 pts) Prove

$$\frac{2}{\pi} \leqslant \frac{\sin x}{x} \leqslant 1 \tag{5}$$

for all $0 \leqslant x \leqslant \pi/2$. (Hint: Show $f(x) = \frac{\sin x}{x}$ is decreasing).

Problem 6. (3 pts) Let $f(x) = \sin 2\,x$.

a) **(1 pt)** Calculate its Taylor polynomial of degree 3 at $x_0 = 0$ with Lagrange remainder;

b) **(2 pts)** Denote this polynomial by $T_3(x)$ (does not include the remainder term). Prove that $|\sin 2x - T_3(x)| < \frac{1}{120}$ for all $-\frac{1}{2} < x < \frac{1}{2}$.