Math 314 A1 Fall 2012 Homework 4

Due Thursday Nov. 1.

5:30pm (Assignment box CAB 3rd floor)

- Related sections in notes: §3.1 – §3.3.

Problem 1. (5 pts) Let $g(x)$ be a continuous function (that is g continuous at all $x \in \mathbb{R}$). Prove that $f(x) = \begin{cases} g(x)\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous for all $x \in \mathbb{R}$ if and only if $g(0) = 0$.

Problem 2. (5 pts) Assume there is $\delta_0 > 0$ such that $h(x) \leqslant f(x) \leqslant g(x)$ for all $x \in (x_0 - \delta_0, x_0 + \delta_0)$. Further assume that h, g are continuous at x_0. What extra condition do we need on h, g to be able to conclude "f is continuous at x_0"? Justify your answer (You need to prove your condition is necessary and sufficient).

Problem 3. (6 pts) Let $f(x)$ be a real function and $x_0 \in \mathbb{R}$. We say L is the left-limit of f at x_0 if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that for all } -\delta < x - x_0 < 0, \qquad |f(x) - L| < \varepsilon. \tag{1}$$

We denote it by

$$\lim_{x \longrightarrow x_0-} f(x) = L; \tag{2}$$

We say L is a right-limit of f at x_0 if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that for all } 0 < x - x_0 < \delta, \qquad |f(x) - L| < \varepsilon. \tag{3}$$

We denote it by

$$\lim_{x \longrightarrow x_0+} f(x) = L. \tag{4}$$

We say f is left continuous if $\lim_{x - x_0 -} f(x) = f(x_0)$; We say f is right continuous if $\lim_{x \longrightarrow x_0+} f(x) = f(x_0)$.

a) **(2 pts)** Give an example of a function $f(x)$ satisfying: Both $\lim_{x \to x_0-} f(x)$ and $\lim_{x \to x_0+} f(x)$ exist but are not equal. Justify your answer.

b) **(2 pts)** Prove that L is the limit of f at x_0 if and only if

$$\lim_{x \longrightarrow x_0-} f(x) = \lim_{x \longrightarrow x_0+} f(x) = L. \tag{5}$$

c) **(2 pts)** Prove that $f(x)$ is continuous at x_0 if and only if $f(x)$ is both left and right continuous at x_0.

Problem 4. (4 pts) Let $f(x): \mathbb{R} \mapsto \mathbb{R}$ satisfy the intermediate value property, that is for any x_1, x_2 and any value s between $f(x_1)$ and $f(x_2)$, there is ξ between x_1, x_2 such that $f(\xi) = s$.

a) **(2 pts)** Is $f(x)$ necessarily continuous? Justify your answer.

b) **(2 pts)** What if we further assume that $\lim_{x \to x_0+} f(x)$ and $\lim_{x \to x_0-} f(x)$ (see Problem 3) both exist for every $x_0 \in \mathbb{R}$?