3. Real Functions and Continuity

A real function is a function with domain $\mathbb{R}$ and range $\mathbb{R}$.

3.1. Limits of Real Functions.

3.1.1. Definition.

Consider a real function $f(x)$. We would like to understand its limiting behavior toward a point x_0, that is, we would like to understand $f(x) \longrightarrow ?$ as $x \longrightarrow x_0$.

Definition 3.1. *We say that a real number L is the limit of f at x_0, denoted $\lim_{x \to x_0} f(x) = L$ (or $f(x) \longrightarrow L$ as $x \longrightarrow x_0$, or $\lim f(x) = L$ as $x \longrightarrow x_0$), if*

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that for all } 0 < |x - x_0| < \delta \Longrightarrow |f(x) - L| < \varepsilon. \tag{3.1}$$

Remark 3.2. Note that the requirement $0 < |x - x_0|$ is important. The limit of f at x_0 has **nothing** to do with whether $f(x_0)$ is defined or not, not to say its value. This is very reasonable: Consider $f(x) = 0$ everywhere except $f(x) = 1$ at $x = 0$. Then clearly we should have $\lim_{x \longrightarrow 0} f(x) = 0$.

Remark 3.3. It is also possible to define $\lim_{x \to x_0} f(x)$ when the domain of f is $E \subseteq \mathbb{R}$ instead of the whole $\mathbb{R}$. In this case we say $\lim_{x \to x_0} f(x) = L$ if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that for all } x \in E \text{ satisfying } 0 < |x - x_0| < \delta \Longrightarrow |f(x) - L| < \varepsilon. \tag{3.2}$$

The following theorems still hold in this general case.

Example 3.4. The following hold:

a) $\lim_{x \longrightarrow x_0} a = a$ for constant function a.

b) $\lim_{x \longrightarrow 1} x^2 = 1$.

Proof. We prove b).

Given any $\varepsilon > 0$, we need to find $\delta > 0$ such that for all $0 < |x - 1| < \delta$, $|x^2 - 1| < \varepsilon$. Since

$$|x^2 - 1| = |x - 1|\,|x + 1| < \delta\,|x + 1| < \delta\,(2 + \delta). \tag{3.3}$$

we see that we need to choose a δ such that $\delta\,(2 + \delta) < \varepsilon$.

Notice that

$$\delta\,(2 + \delta) < \varepsilon \Longleftrightarrow \delta^2 + 2\,\delta + 1 < \varepsilon + 1 \Longleftrightarrow |\delta + 1| < \sqrt{\varepsilon + 1} \Longleftrightarrow -1 - \sqrt{\varepsilon + 1} < \delta < \sqrt{\varepsilon + 1} - 1. \tag{3.4}$$

Thus for any $\varepsilon > 0$, if we take $\delta = \frac{\sqrt{\varepsilon + 1} - 1}{2}$, then for $0 < |x - 1| < \delta$, $|x^2 - 1| < \varepsilon$. $\square$

Lemma 3.5. *A function can have at most one limit at a given point.*

Proof. We prove by contradiction. Assume that $f(x) \longrightarrow L_1$ and $f(x) \longrightarrow L_2$ as $x \longrightarrow x_0$. Take $\varepsilon = |L_1 - L_2|/2$. From the first limit we conclude that there is $\delta_1 > 0$ such that for all $0 < |x - x_0| < \delta_1$, $|f(x) - L_1| < \varepsilon$; Similarly from the second limit we conclude that there is $\delta_2 > 0$ such that for all $0 < |x - x_0| < \delta_2$, $|f(x) - L_2| < \varepsilon$.

Now take $\delta = \min\{\delta_1, \delta_2\}$. For every x such that $|x - x_0| < \delta$, we have at the same time $|f(x) - L_1| < \varepsilon$ and $|f(x) - L_2| < \varepsilon$. This leads to

$$|L_1 - L_2| = |(f(x) - L_1) - (f(x) - L_2)| \leqslant |f(x) - L_1| + |f(x) - L_2| < 2\,\varepsilon = |L_1 - L_2|. \tag{3.5}$$

Contradiction. □

Theorem 3.6. $\lim_{x \longrightarrow x_0} f(x) = L$ *if and only if for every sequence* $x_n \longrightarrow x_0$ *with* $x_n \neq x_0$ *for all* $n \in \mathbb{N}$, $\lim_{n \longrightarrow \infty} f(x_n) = L$.

Proof. We prove "if" ($\Longleftarrow$) and then "only if" ($\Longrightarrow$),

- "If". We need to show if for every sequence $x_n \longrightarrow x_0$ with $x_n \neq x_0$ for all $n \in \mathbb{N}$, $\lim_{n \longrightarrow \infty} f(x_n) = L$ then $\lim_{x \longrightarrow x_0} f(x) = L$. We prove by contradiction.

  Assume the contrary. That is assume $f(x)$ does not have limit L as $x \longrightarrow x_0$. Checking Definition 3.1 we see that "$f(x)$ does not have limit L as $x \longrightarrow x_0$" means

  $$\exists \varepsilon_0\ \forall \delta\ \exists x \text{ with } 0 < |x - x_0| < \delta, \qquad |f(x) - L| > \varepsilon_0. \tag{3.6}$$

  Take $\delta = 1$. We have x_1 with $0 < |x_1 - x_0| < 1$, such that

  $$|f(x_1) - L| > \varepsilon_0; \tag{3.7}$$

  Next take $\delta = 1/2$. We have x_2 with $0 < |x_2 - x_0| < 1/2$, such that

  $$|f(x_2) - L| > \varepsilon_0; \tag{3.8}$$

  Continue doing this, we obtain a sequence $\{x_n\}$ satisfying $0 < |x_n - x_0| < 1/n$,

  $$|f(x_n) - L| > \varepsilon_0. \tag{3.9}$$

  Clearly $x_n \neq x_0$ for all $n \in \mathbb{N}$ by construction. To reach contradiction, we prove

  - $x_n \longrightarrow x_0$. For any $\varepsilon > 0$, choose natural number $N > 1/\varepsilon$. Then for every $n > N$, we have

    $$|x_n - x_0| < 1/N < \varepsilon. \tag{3.10}$$

  - $f(x_n) \not\longrightarrow L$. Take $\varepsilon = \varepsilon_0$. Since by construction $|f(x_n) - L| > \varepsilon_0$ for all $n \in \mathbb{N}$, there does not exists $N \in \mathbb{N}$ such that $n > N \Longrightarrow |f(x_n) - L| < \varepsilon$.

  Thus we have shown that there is a sequence $x_n \longrightarrow x_0$ with $x_n \neq x_0$ for all $n \in \mathbb{N}$ but $\lim_{n \longrightarrow \infty} f(x_n) = L$ does not hold. Contradiction.

- "Only if". We need to show that if $\lim_{x \longrightarrow x_0} f(x) = L$ then for every sequence $x_n \longrightarrow x_0$ with $x_n \neq x_0$ for all $n \in \mathbb{N}$, $\lim_{n \longrightarrow \infty} f(x_n) = L$. We prove directly this time.

  Let $\{x_n\}$ be a sequence satisfying $x_n \longrightarrow x_0$ with $x_n \neq x_0$ for all $n \in \mathbb{N}$. Given any $\varepsilon > 0$, since $\lim_{x \longrightarrow x_0} f(x) = L$ there is $\delta > 0$ such that $0 < |x - x_0| < \delta \Longrightarrow |f(x) - L| < \varepsilon$.

  Now for this δ, we have $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - x_0| < \delta$ because $x_n \longrightarrow x_0$. Thus for all $n > N$,$|f(x_n) - L| < \varepsilon$.

Putting the above together, we see that for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$, $|f(x_n) - L| < \varepsilon$ which means $\lim_{n \longrightarrow \infty} f(x_n) = L$. $\square$

The assumptions in the above theorem can be weakened to

Theorem 3.7. *$\lim_{x \longrightarrow x_0} f(x)$ exists if and only if for every sequence $x_n \longrightarrow x_0$ with $x_n \neq x_0$ for all $n \in \mathbb{N}$, $\lim_{n \longrightarrow \infty} f(x_n)$ exists.*

Proof. Left as exercise. (Hint: If all such limits exist, then they must be the same number). $\square$

3.1.2. Properties.

Theorem 3.8. *If $\lim_{x \to x_0} f(x) = L, \lim_{x \to x_0} g(x) = M$, then*

$$\lim_{x \to x_0} (f \pm g)(x) = L \pm M, \qquad \lim_{x \to x_0} (f g)(x) = L M, \tag{3.11}$$

and, if furthermore $M \neq 0$,

$$\lim_{x \to x_0} \left(\frac{f}{g}\right)(x) = \frac{L}{M}. \tag{3.12}$$

Proof. Thanks to Theorem 3.6, these claims directly follow from corresponding claims for sequences. $\square$

One can clearly obtain Squeeze and Comparison theorems:

Theorem 3.9. (Squeeze) *If $h(x) \leqslant f(x) \leqslant g(x)$, $\lim_{x \to x_0} h(x) = \lim_{x \to x_0} g(x) = L$, then $\lim_{x \to x_0} f(x) = L$.*

Example 3.10. Prove that $\lim_{x \longrightarrow 0} x \sin(1/x) = 0$.

Proof. We take $h(x) = -|x|$, $g(x) = |x|$, $f(x) = x \sin(1/x)$. Clearly $h(x) \leqslant f(x) \leqslant g(x)$. Thus all we need to show is $h(x)$, $g(x) \longrightarrow 0$. Further observe that $h(x) = -g(x)$, by Theorem 3.8 and the fact that $\lim_{x \longrightarrow 0} (-1) = -1$, we have $\lim_{x \longrightarrow 0} h(x) = -\lim_{x \longrightarrow 0} g(x)$.

Therefore all we need to show is $\lim_{x \longrightarrow 0} |x| = 0$. For every $\varepsilon > 0$, take $\delta = \varepsilon$, we have for all $|x - 0| < \delta$,

$$||x| - 0| = |x| < \delta = \varepsilon. \tag{3.13}$$

Thus ends the proof. $\square$

Theorem 3.11. (Comparison) *If $h(x) \leqslant f(x) \leqslant g(x)$, and $\lim_{x \to x_0} h(x) = L_1$, $\lim_{x \to x_0} f(x) = L_2$, $\lim_{x \to x_0} g(x) = L_3$, then $L_1 \leqslant L_2 \leqslant L_3$.*

One can similarly define divergence to $\pm\infty$:

Definition 3.12. *We say $\lim_{x \to x_0} f(x) = +\infty$ if*

$$\forall M \in \mathbb{R}\ \exists \delta > 0 \text{ such that for all } 0 < |x - x_0| < \delta, \qquad f(x) > M. \tag{3.14}$$

$\lim_{x \to x_0} f(x) = -\infty$ is defined similarly.

Example 3.13. Prove that $\ln(1/|x|) \longrightarrow \infty$ as $x \longrightarrow 0$.

Proof. For any $M \in \mathbb{R}$, we need to find $\delta > 0$ such that when $0 < |x| < \delta$, $\ln(1/|x|) > M$. It is clear that we can take $\delta = e^{-M-1}$. $\square$

Remark 3.14. Theorems 3.6, 3.8 as well as the Squeeze and Comparison theorems can all be generalized to the cases $L = \pm\infty$. The proofs are left as exercises.

3.1.3. Limits at infinity.

Definition 3.15. *We say $\lim_{x \longrightarrow \infty} f(x) = L$ if for every $\varepsilon > 0$ there is some $M \in \mathbb{R}$ such that $x > M \Longrightarrow |f(x) - L| < \varepsilon$. The limit $\lim_{x \longrightarrow -\infty} f(x) = L$ is defined similarly.*

Remark 3.16. One can also define $\lim_{x \longrightarrow \infty} f(x) = \infty$ etc. So essentially, we can discuss $\lim_{x \longrightarrow x_0} f(x) = L$ for **extended real numbers** x_0, L and obtain generalized versions of Theorems 3.6, 3.8 as well as the Squeeze and Comparison theorems. The detailed definitions for these general situations are left as exercise.

Example 3.17. Find and prove the limit (only discuss $x > 0$)

$$\lim_{x \longrightarrow \infty} \left(\sqrt{x^2 + 2x} - x \right). \tag{3.15}$$

Solution.

The function is quite complicated and the limit is not obvious. Thus we first try to simplify:

$$\sqrt{x^2 + 2x} - x = \frac{\left(\sqrt{x^2+2x} - x\right)\left(\sqrt{x^2+2x} + x\right)}{\left(\sqrt{x^2+2x} + x\right)} = \frac{2x}{\sqrt{x^2+2x} + x} = \frac{2}{\sqrt{1 + \frac{2}{x}} + 1}. \tag{3.16}$$

By Theorem 3.8 we see that all we need to do is to find the limits of the numerator and denominator. Clearly

$$\lim_{x \longrightarrow \infty} 2 = 2. \tag{3.17}$$

On the other hand we show $\lim_{x \longrightarrow \infty} \sqrt{1 + 2/x} = 1$. For any $\varepsilon > 0$, take $M = 1/\varepsilon$. Then for all $x > M$, we have $2/x < 2\varepsilon$ and consequently

$$0 < \sqrt{1 + 2/x} - 1 < \sqrt{1 + 2\varepsilon} - 1 < \sqrt{(1+\varepsilon)^2} - 1 = \varepsilon. \tag{3.18}$$

Therefore $\left|\sqrt{1+2/x} - 1\right| < \varepsilon$ for all $x > M$. We have proved $\lim_{x \longrightarrow \infty} \left(\sqrt{1 + 2/x} + 1\right) = 2 \neq 0$.

Now we can apply Theorem 3.8 once more to conclude

$$\lim_{x \longrightarrow \infty} \left(\sqrt{x^2 + 2x} - x\right) = \lim_{x \longrightarrow \infty} \frac{2}{\sqrt{1 + \frac{2}{x}} + 1} = 1. \tag{3.19}$$

Remark 3.18. One can also use Comparison theorem as follows:

$$1 = \frac{2x}{2x} < \frac{2x}{\sqrt{x^2 + 2x} + x} < \frac{2x}{2x + 1} \tag{3.20}$$

and then prove $\lim_{x \longrightarrow \infty} \frac{2x}{2x+1}$. Usually there are many ways to solve a problem in mathematical analysis. Be creative.

3.2. Continuity.

3.2.1. Definition.

Definition 3.19. (Continuity) *A function $f(x): \mathbb{R} \mapsto \mathbb{R}$ is continuous at x_0 if*

1. $\lim_{x \longrightarrow x_0} f(x)$ *exists;*
2. $f(x_0) = \lim_{x \longrightarrow x_0} f(x)$.

Or equivalently, if

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that for all } |x - x_0| < \delta, \qquad |f(x) - f(x_0)| < \varepsilon. \tag{3.21}$$

If $f(x)$ is continuous at every $x \in E \subseteq \mathbb{R}$, we say $f(x)$ is continuous on E. In particular, we simply say $f(x)$ is continuous when $E = \mathbb{R}$.

Remark 3.20. Note that compared with Definition 3.1 (limit of function), we do not require $0 < |x - x_0|$ anymore.

Remark 3.21. Continuity can be similarly defined if the domain of f is only a subset of $\mathbb{R}$.

Example 3.22. Let $f_1(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Prove that it is continuous at $x = 0$.

Proof. Take any $\varepsilon > 0$ we need to find $\delta > 0$ such that for every $|x - 0| < \delta$, $|f_1(x) - 0| < \varepsilon$. We take $\delta = \left(\frac{1}{\log(1/\varepsilon)}\right)^{1/2}$. To show that this δ works, we need to discuss two cases (be careful here!!):

- $x = 0$. In this case $|0 - 0| < \varepsilon$;
- $x \neq 0$. In this case $|f_1(x) - 0| = e^{-1/x^2} < e^{-1/\delta^2} = \varepsilon$. □

Example 3.23. Let $f_2(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & |x| \geqslant 1 \end{cases}$. Prove that it is continuous at $x = 1$.

Proof. Take any $\varepsilon > 0$ we need to find $\delta > 0$ such that for every $|x - 1| < \delta$, $|f_2(x) - 0| < \varepsilon$. Take $\delta = \ln(1/\varepsilon)$, then for every $|x - 1| < \delta$ there are two cases:

- $x \geqslant 1$. In this case $|f_2(x) - f_2(1)| = 0 < \varepsilon$.
- $x < 1$. In this case

$$|f_2(x) - f_2(1)| = e^{-\frac{1}{(1-x)(1+x)}} < e^{-\frac{1}{\delta}} = \varepsilon. \tag{3.22}$$

Thus ends the proof. □

Remark 3.24. When showing continuity by definition, the key is to find a formula for δ. Usually this formula cannot be revealed until enough simplification has been done to $f(x) - f(x_0)$. Therefore it is a good idea to first write done:

> For any given $\varepsilon > 0$, we take $\delta =$ (leave blank for now), then for every $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| = \ldots$.(simplifications).

and then fill in the black for δ when enough simplification is done.

Remark 3.25. It is important to understand that saying $f(x)$ is continuous at x_0 has two layers of meanings, first, $\lim_{x \to x_0} f(x)$ exists, second, this limit is exactly $f(x_0)$. Correspondingly, $f(x)$ is **not** continuous at x_0[3.1] if either

- $\lim_{x \to x_0} f(x)$ does not exist, or
- $\lim_{x \to x_0} f(x)$ exists but does not equal $f(x_0)$.

Example 3.26. Construct examples of functions that are not continuous.

- We start from the simplest one: $f(x) = \begin{cases} \sin x & x \neq 0 \\ 2 & x = 0 \end{cases}$ is not continuous at 0.
- $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not continuous at $x = 0$.
- The Dirichlet function $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$ is not continuous anywhere.

Theorem 3.27. *f is continuous at x_0 if and only if for every sequence $x_n \longrightarrow x_0$, $f(x_n) \longrightarrow f(x_0)$.*

Proof. It follows immediately from Definition 3.19 and Theorem 3.6. □

3.2.2. Properties.

Theorem 3.28. *Let f, g be functions continuous at x_0. Then $f \pm g$, $f g$ are continuous at x_0. Furthermore if $g(x_0) \neq 0$, f/g is also continuous at x_0.*

Proof. Left as exercise. □

Example 3.29. (Continuity of everyday functions) The following functions are continuous everywhere.

a) Polynomials $P(x) = a_n x^n + \cdots + a_1 x + a_0$.

b) The exponential function e^x.

c) The functions $\sin x$, $\cos x$.

Proof. The proofs of b), c) involve something beyond 314. So we only prove a) here. Thanks to Theorem 3.28, all we need to show is that the monomial x^k is continuous everywhere.

First consider the case $x_0 = 0$. In this case for every $\varepsilon > 0$, take $\delta = \varepsilon^{1/k}$. Then for all $|x - 0| < \delta$ we have $|x^k - 0^k| < \delta^k = \varepsilon$.

Next consider the case $x_0 > 0$. In this case for every $\varepsilon > 0$, take $\delta = \min\left(x_0, \varepsilon / \left(k\, 2^{k-1} x_0^{k-1}\right)\right)$, then for all $|x - x_0| < \delta$, we have $|x| < 2\, x_0$ and $|x^k - x_0^k| = \left|(x - x_0)\left(x^{k-1} + x^{k-2} x_0 + \cdots + x_0^{k-1}\right)\right| = k\, 2^{k-1} x_0^{k-1} \delta = \varepsilon$.

The case $x_0 < 0$ is similar and left as exercise. □

Example 3.30. (Rational functions) A rational function is the ratio of two polynomials: $f(x) = \frac{p(x)}{q(x)}$ where p, q are polynomials. Since p, q are continuous at every $x \in \mathbb{R}$, $f(x) = \frac{p(x)}{q(x)}$ is continuous at all $x \in \mathbb{R}$ such that $q(x) \neq 0$.

If $q(x_0) = 0$, then there are two cases.

1. If after cancelling all common factors, $f(x) = \frac{p_1(x)}{q_1(x)}$ with $q_1(x_0) \neq 0$, then $f(x)$ is continuous at x_0.
2. If after cancellation we still have $q_1(x) = 0$, then the limit does not exist.

For example, consider

$$f(x) = \frac{x^2 - 3\,x + 2}{x^2 - 4}. \tag{3.23}$$

3.1. Keep in mind that the best way to understand a definition in mathematical analysis is to construct examples of those that do not satisfy the definition. Once you can construct one example for every single way of possible violation of the requirements of the definition, you can say

Factorize:

$$x^2-3\,x+2=(x-1)\,(x-2);\qquad x^2-4=(x+2)\,(x-2), \tag{3.24}$$

so

$$f(x)=\frac{x-1}{x+2}. \tag{3.25}$$

By Theorem 3.28, $f(x)$ is continuous at every $x\in\mathbb{R}$, with $x\neq-2$. At $x=-2$, the limit doesn't exist since $x-1\neq 0$.

Example 3.31. Ratios involving functions other than polynomials are subtle to deal with. For example, although no cancellation can be made for $f(x)=\frac{\sin x}{x}$, it is indeed continuous everywhere, even at $x=0$.

A very useful property of continuity is regarding composite functions.

Definition 3.32. *Let $f\colon A\mapsto B$, $g\colon B\mapsto\mathbb{R}$ where A,B are subsets of $\mathbb{R}$. Then the composition function $g\circ f$ is defined as*

$$(g\circ f)(x)=g(f(x)). \tag{3.26}$$

Example 3.33. The function e^{-1/x^2} is a composite function: $e^{-1/x^2}=g\circ f$ with $f(x)\colon\{x\in\mathbb{R}\colon x\neq 0\}\mapsto\mathbb{R}$ defined as $f(x)=-1/x^2$, and $g(x)=e^x$.

Obviously there are more than one way to write a function into composite functions. For example we also have $e^{-1/x^2}=g_1\circ f_1$ with $f_1=1/x^2$ and $g_1=e^{-x}$. We can even write it into composition of more than two functions: $e^{-1/x^2}=h_1\circ h_2\circ h_3\circ h_4$ with

$$h_1(x)=e^x,\quad h_2(x)=-x,\quad h_3(x)=x^2,\quad h_4(x)=1/x. \tag{3.27}$$

It is important to understand that in general $f\circ g\neq g\circ f$. For example $g=e^x$ and $f=x^2$.

Theorem 3.34. (Continuity of composite functions) *Let $f\colon A\mapsto B$, $g\colon B\mapsto\mathbb{R}$ If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g\circ f$ is continuous at x_0.*

Proof. Given any $\varepsilon>0$, there is $\delta_1>0$ such that for all $|y-f(x_0)|<\delta_1$, $|g(y)-g(f(x_0))|<\varepsilon$; On the other hand, the continuity of f gives: there is $\delta>0$ such that for all $|x-x_0|<\delta$, $|f(x)-f(x_0)|<\delta_1$. Combine these, we see that $|g(f(x))-g(f(x_0))|<\varepsilon$. □

Example 3.35. Show that the following functions are continuous on $\mathbb{R}$.

a) $f_1(x)=\begin{cases} e^{-1/x^2} & x\neq 0\\ 0 & x=0\end{cases}$.

b) $f_2(x)=\begin{cases} e^{-\frac{1}{1-x^2}} & |x|<1\\ 0 & |x|>1\end{cases}$.

Proof. We only prove a). The proof for b) is similar.

Note that $f_1(x)=f\circ g$ with $f=e^x$, $g=-1/x^2$. We know that f is continuous everywhere, and g is continuous everywhere except at $x=0$. Therefore $f_1(x)$ is continuous at every $x\neq 0$. Since we have already shown in a previous example that $f_1(x)$ is continuous at 0 too, we conclude that $f_1(x)$ is continuous everywhere. □

Remark 3.36. We will see in future lectures that f_1,f_2 in fact are infinitely differentiable on $\mathbb{R}$, which may be counter-intuitive.

3.3. Further Discussions about Limit and Continuity.

3.3.1. Continuous functions fit intuition.

Intuitively, a function $f: \mathbb{R} \mapsto \mathbb{R}$ is simply a curve running from far far left to far far right. One naturally expect it to have the following properties:

- Intermediate value. If $f(x_1) = a$ and $f(x_2) = b$, then f takes all intermediate values between a, b;
- Maximum and minimum. Between any $a, b \in \mathbb{R}$, that is over any finite interval, f reaches its maximum and minimum.

Remark 3.37. It is obvious that one should not expect f to reach maximum or minimum if the interval's size is infinite since the function may be unbounded.

Example 3.38. (Not continuous is bad) One can easily construct examples of discontinuous functions that do not have the above good properties. For example, the function $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leqslant 0 \end{cases}$ does not satisfy the intermediate value property; The function $f(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 1 & |x| > 1 \end{cases}$ does not reach its minimum.

In the following we will show that when f is continuous, our intuition can be justified, although some minor assumptions need to be added.

Theorem 3.39. (Intermediate value) *Let $f(x)$ be continuous on the closed interval $[a, b]$. Then for every $s \in [f(a), f(b)]$ (or $[f(b), f(a)]$ if $f(b) \leqslant f(a)$), there is $\xi \in [a, b]$ such that $f(\xi) = s$.*

Proof. Assume $f(a) \leqslant f(b)$. Let $x_1 = a$, $y_1 = b$. We have $f(x_1) \leqslant s \leqslant f(y_1)$. Define x_n, y_n as follows: Suppose x_k, y_k are known. Then set

- $x_{k+1} = \frac{x_k + y_k}{2}$, $y_{k+1} = y_k$ if $f\left(\frac{x_k + y_k}{2}\right) \leqslant s$;
- $y_{k+1} = \frac{x_k + y_k}{2}$, $x_{k+1} = x_k$ if $f\left(\frac{x_k + y_k}{2}\right) > s$.

This way we obtain two sequences $\{x_n\}, \{y_n\}$ satisfying $x_{n+1} \geqslant x_n$, $y_{n+1} \leqslant y_n$ and furthermore

$$f(x_n) \leqslant s, \qquad f(y_n) \geqslant s, \qquad |x_n - y_n| \leqslant \frac{|x_{n-1} - y_{n-1}|}{2}. \tag{3.28}$$

This means $x_n - y_n \longrightarrow 0$.

Now as $\{x_n\}$ is increasing and bounded above, $\lim_{n \longrightarrow \infty} x_n$ exists; On the other hand, since $\{y_n\}$ is decreasing and bounded below, $\lim_{n \rightarrow \infty} y_n$ exists. Because $x_n - y_n \longrightarrow 0$ the two limits are equal. Call it ξ.

Because f is continuous on $[a, b]$, together with comparison theorem we have $\lim_{n \longrightarrow \infty} f(x_n) = f(\xi) \leqslant s$; Similarly we have $\lim_{n \longrightarrow \infty} f(y_n) = f(\xi) \geqslant s$. Combining $f(\xi) \geqslant s$ and $f(\xi) \leqslant s$ we conclude $f(\xi) = s$. $\square$

Remark 3.40. A few remarks.

- There may be more than one ξ satisfying $f(\xi) = s$. For example consider $f(x) = \sin x$.
- It is not enough to assume $f(x)$ to be continuous over the open interval (a, b). For example, take the Heaviside function $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leqslant 0 \end{cases}$. $f(x)$ is continuous over $(0, 1)$ but it does not take any intermediate value between $f(0) = 0$ and $f(1) = 1$.

Example 3.41. One application of the intermediate value theorem is to find zeroes of functions. For example, one can prove:

> **Theorem.** Any odd-degree polynomial has at least one real root.

Proof. Denote the polynomial as

$$P(x) = a_{2k-1} x^{2k-1} + a_{2k-2} x^{2k-2} + \cdots + a_1 x + a_0, \qquad a_{2k-1} \neq 0 \tag{3.29}$$

By the Intermediate Value Theorem 3.39 all we need to do is to find $a, b \in \mathbb{R}$ such that $f(a) < 0, f(b) > 0$. Write

$$P(x) = a_{2k-1} x^{2k-1} \left[1 + \frac{a_{2k-2}}{a_{2k-1}} \frac{1}{x} + \frac{a_{2k-3}}{a_{2k-1}} \frac{1}{x^2} + \cdots + \frac{a_1}{a_{2k-1}} \frac{1}{x^{2k-2}} + \frac{a_0}{a_{2k-1}} \frac{1}{x^{2k-1}} \right]. \tag{3.30}$$

Then we can prove that[3.2]

$$1 + \frac{a_{2k-2}}{a_{2k-1}} \frac{1}{x} + \frac{a_{2k-3}}{a_{2k-1}} \frac{1}{x^2} + \cdots + \frac{a_1}{a_{2k-1}} \frac{1}{x^{2k-2}} + \frac{a_0}{a_{2k-1}} \frac{1}{x^{2k-1}} \longrightarrow 1 \tag{3.31}$$

as $x \longrightarrow \infty$ or $x \longrightarrow -\infty$. Therefore there is $M > 0$ such that for all $|x| > M$, the sign of $P(x)$ is the same as that of $a_{2k-1} x^{2k-1}$. Thus there are $a, b \in \mathbb{R}$ such that $f(a) < 0, f(b) > 0$.

To show the maximum and minimum property, we first need the following boundedness result.

Theorem 3.42. (Boundedness) *Let $f(x)$ be continuous on $[a, b]$ for $a, b \in \mathbb{R}$, then $f(x)$ is bounded on $[a, b]$. That is there is $M \in \mathbb{R}$ such that $|f(x)| \leqslant M$ for all $x \in [a, b]$.*

Proof. We prove by contradiction. Assume that

$$\forall M \in \mathbb{R}\ \exists x \in [a, b], \qquad |f(x)| > M. \tag{3.32}$$

Taking $M = 1, 2, 3, \ldots$ we obtain a sequence $\{x_n\}$ satisfying $|f(x_n)| > n$. Now apply Bolzano-Weierstrass to $\{x_n\}$ we obtain a converging subsequence $x_{n_k} \longrightarrow \xi$. As f is continuous, we conclude $f(x_{n_k})$ is a bounded sequence; But at the same time $|f(x_{n_k})| > n_k$ implies $f(x_{n_k})$ is an unbounded sequence. Contradiction. □

Theorem 3.43. (Maximum and minimum) *Let $f(x)$ be continuous on $[a, b]$ for $a, b \in \mathbb{R}$. Then f reaches both maximum and minimum, that is there are $x_{\max}, x_{\min} \in [a, b]$ such that*

$$f(x_{\min}) \leqslant f(x) \leqslant f(x_{\max}) \qquad \forall x \in [a, b]. \tag{3.33}$$

Proof. From the boundedness theorem 3.42 we know that there is M such that $|f(x)| \leqslant M$ for all $x \in [a, b]$. Let $s = \sup_{x \in [a, b]} f(x) := \sup \{f(x) : x \in [a, b]\}$. Then $s \in \mathbb{R}$ and there is a sequence x_n such that $f(x_n) \longrightarrow s$. As $x_n \in [a, b]$ the sequence is bounded so we can apply Bolzano-Weierstrass to obtain a converging subsequence $x_{n_k} \longrightarrow x_{\max} \in [a, b]$. The continuity of f at ξ now gives

$$f(x_{\max}) = \lim_{k \longrightarrow \infty} f(x_{n_k}) = s. \tag{3.34}$$

That is f attains maximum at $x_{\max}$. The other half of the theorem can be proved similarly. □

3.3.2. Continuity of inverse functions.

Theorem 3.44. *A continuous real function $f(x): A \mapsto B := f(A)$ has an inverse if and only if $f(x)$ is strictly increasing, that is $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, or strictly decreasing, that is $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.*

Proof.

- "If". If $f(x)$ is strictly increasing, then f is one-to-one. As $B = f(A)$ f is also onto. So there is an inverse. Similar for strictly decreasing.
- "only if". We try to show that if f is one-to-one then it has to be either strictly increasing or strictly decreasing. Take any $x_1 < x_2$. There are two cases:
  - $f(x_1) < f(x_2)$. We show that f must be strictly increasing.
    1. First show for all $x > x_2$, we have $f(x) > f(x_2)$. Assume the contrary. Then either there is $x > x_2$ such that $f(x) = f(x_2)$, which already leads to contradiction to the fact that f is one-to-one, or there is $x > x_2$ such that $f(x) < f(x_2)$. Now compare $f(x)$ and $f(x_1)$. If $f(x) \leqslant f(x_1)$, then by intermediate value theorem there is $\xi \in [x_2, x]$ (thus $\xi \neq x_1$) with $f(\xi) = f(x_1)$, contradiction. If $f(x) > f(x_1)$, then again by intermediate value theorem there is $\xi \in [x_1, x_2]$ such that $f(\xi) = f(x)$. Contradiction.

3.2. If it's in an exam, you have to prove it in detail.

2. Similarly one can show that for all $x < x_1$, $f(x) < f(x_1)$.
3. Next we show that for all $x \in (x_1, x_2)$, $f(x_1) < f(x) < f(x_2)$. Assume not, then either there is x with $f(x) \geqslant f(x_2)$ or there is x with $f(x) \leqslant f(x_1)$. Either case, we can apply intermediate value theorem to reach a contradiction.
4. Now we show that for all $x_3 < x_4$, $f(x_3) < f(x_4)$. There are several cases.
   a. $x_3 \in$ one of $(-\infty, x_1), [x_1, x_2], (x_2, \infty)$ while $x_4 \in$ another. Then from what we have shown $f(x_3) < f(x_4)$.
   b. x_3, $x_4 \in$ the same one of $(-\infty, x_1)$, $[x_1, x_2]$, (x_2, ∞). We prove for the case x_3, $x_4 \in (-\infty, x_1)$, other cases are similar. Assume $f(x_3) < f(x_4)$ does not hold. If $f(x_3) = f(x_4)$, we already have contradiction; If $f(x_3) > f(x_4)$, then we have $f(x_3) > f(x_4) < f(x_1)$, application of intermediate value theorem leads to contradiction.

- $f(x_1) > f(x_2)$. Similarly we can show that in this case f must be strictly decreasing. □

Remark 3.45. The above proof is kind of messy. The idea is however simple: By intermediately value theorem, whenever we have $x_1 < x_2 < x_3$ with $f(x_1) \leqslant f(x_2) \geqslant f(x_3)$ or $f(x_1) \geqslant f(x_2) \leqslant f(x_3)$ then f cannot be one-to-one. However I haven't been able to find a simple proof yet.

Theorem 3.46. *Let $f: A \mapsto B$ be continuous, onto, and strictly increasing (strictly decreasing). Then the inverse $g: B \mapsto A$ is continuous, onto, and strictly increasing (strictly decreasing).*

Proof. That g is onto is clear. To see that g is strictly increasing, take $y_1 < y_2$. Then we have

$$f(g(y_1)) = y_1 < y_2 = f(g(y_2)). \tag{3.35}$$

As f is strictly increasing, we must have $g(y_1) < g(y_2)$ since otherwise we would have $f(g(y_1)) \geqslant f(g(y_2))$.

Now we show that g is continuous. Assume the contrary, that is assume g is not continuous at some y_0. Then by definition there is $\varepsilon_0 > 0$ such that there is $y_n \longrightarrow y_0$ but $|g(y_n) - g(y_0)| > \varepsilon_0$. Now let

$$\delta_0 = \min\{f(g(y_0)) - f(g(y_0) - \varepsilon_0), f(g(y_0) + \varepsilon_0) - f(g(y_0))\} > 0, \tag{3.36}$$

Note that $\delta_0 > 0$ is because f is strictly increasing, we obtain

$$|y_{n_k} - y_0| = |f(g(y_{n_k})) - f(g(y_0))| > \delta_0 > 0. \tag{3.37}$$

But this contradicts $y_{n_k} \longrightarrow y_0$.

Thus ends the proof. □

Example 3.47. (Log) We see that $\ln x$ is continuous for $x > 0$.