Math 314 A1 Fall 2012 Homework 2

Due Thursday Oct. 4

5:30pm (Assignment box CAB 3rd floor)

- Related sections in notes: §0.3.3 – §1.3.

Problem 1. (6 pts) Let $f: X \mapsto Y$ be a function. Decide which of the following are true. Prove the true one(s) and provide a counterexample for the false one(s).

a) **(2 pts)** f is one-to-one if and only if $f(A \cap B) = f(A) \cap f(B)$ for all subsets A, B of X.

b) **(2 pts)** f has an inverse function if and only if $f(A \cap B) = f(A) \cap f(B)$ for all subsets A, B of X.

c) **(2 pts)** f has an inverse function if and only if for every $A \subseteq X$ and every $S \subseteq Y$, $f^{-1}(f(A)) = A$ and $f(f^{-1}(S)) = S$.

Problem 2. (4 pts) Let $x_n = n^a$ for $a \in \mathbb{R}$. Prove that $\{x_n\}$ converges to 0 when $a < 0$ and diverges when $a > 0$. Then determine whether the following sequences are convergent or not. If convergent find the limit. Justify your answers.

$$x_n = \frac{100\,n^2 - 2\,n^4}{n^4 + 3\,n}, \quad y_n = \sqrt{n+1} - \sqrt{n-1} \tag{1}$$

Problem 3. (3 pts) Let $\{x_n\}$ be a sequence. Suppose that there is $0 < r < 1$ such that $|x_{n+1} - x_n| \leqslant r^n$ for all $n \in \mathbb{N}$. Prove that $x_n \longrightarrow x$ for some $x \in \mathbb{R}$.

Problem 4. (3 pts) Let $x_0 \in \mathbb{R}$ be an arbitrary real number and define x_n through

$$x_n = \frac{x_{n-1}}{3} + 1. \tag{2}$$

Does the sequence converge? If so find the limit. Justify your answer.

Problem 5. (4 pts) Let $0 < y_1 < x_1$ and set

$$x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \sqrt{x_n\, y_n}, \qquad n \in \mathbb{N}. \tag{3}$$

a) **(1 pt)** Prove that $0 < y_n < x_n$ for all $n \in \mathbb{N}$;

b) **(1 pt)** Prove that y_n is increasing and bounded above, and x_n is decreasing and bounded below;

c) **(1 pt)** Prove that $0 < x_{n+1} - y_{n+1} < (x_1 - y_1)/(2^n)$ for all $n \in \mathbb{N}$;

d) **(1 pt)** Prove that $\lim_{n \longrightarrow \infty} x_n$, $\lim_{n \longrightarrow \infty} y_n$ both exist are are equal.