1. $\mathbb{R}$

In this and the next section we are going to study the properties of sequences of real numbers.

Definition 1.1. (Sequence) *A sequence is a function with domain* $\mathbb{N}$.

Example 1.2. A sequence of real numbers is a function with domain $\mathbb{N}$ and range $\mathbb{R}$. We will first construct real numbers using sequences of rational numbers (functions with domain $\mathbb{N}$ and range $\mathbb{Q}$).

Notation. *Often when denoting a sequence, we write* n *not as an argument but as a subscript. That is if* $f: \mathbb{N} \mapsto \mathbb{R}$ *is a real sequence, we usually denote it as* f_n *instead of* $f(n)$.

1.1. Construction of Real Numbers.

We present the main ingredients of construction of real numbers, without providing all the proofs. The important thing here is to understand

1. Why do we need to "construct" $\mathbb{R}$.
2. What are the obstacles we need to overcome during this construction.

1.1.1. Why.

Everyone knows:

- Integers $\mathbb{Z}$ are needed because one cannot subtract within natural numbers $\mathbb{N}$;
- Rationals $\mathbb{Q}$ are needed because one cannot divide within integers $\mathbb{Z}$;
- Complex numbers $\mathbb{C}$ are needed because one cannot solve algebraic equations within real numbers $\mathbb{R}$.

One may say that we need $\mathbb{R}$ because we would like to solve equations like $x^2 - 2 = 0$. But if we don't know anything about $\mathbb{R}$ or $\mathbb{C}$, can we really tell that $x^2 - 2 = 0$ and $x^2 + 2 = 0$ are different types of equations? Not likely.[1.1]

> **Fact.** Note that if we start from $\mathbb{Q}$ and add to it all possible roots of algebraic equations[1.2], as well as linear combinations of these numbers, we still won't get $\mathbb{C}$. To see why, we can either follow the hard way, like proving directly e is transcendental – not root to any algebraic equations; Or follow the easy way, like reading the section about countable/uncountable sets in any analysis book.

What really makes $\mathbb{R}$ necessary is that we would like to "take limit". To see this, we first need an example showing that, indeed, the limit of a sequence of numbers in $\mathbb{Q}$ may not be in $\mathbb{Q}$.

Lemma 1.3. *There is a sequence of rational numbers converging to the irrational number* $\sqrt{2}$.

Proof. (Problematic proof!)

As in the first lecture we have already shown that $\sqrt{2}$ is not rational, we only need a sequence of rational numbers converging to $\sqrt{2}$. Let $p_1 = 1$, $p_2 = 1.4$, $p_3 = 1.41...$ Each p_{n+1} is determined as follows: Let $p_n = 1.41...a_{n-1}$. Then $p_{n+1} = 1.41...a_{n-1}a_n = 1 + 4 \cdot 10^{-1} + 1 \cdot 10^{-2} + \cdots + a_{n-1}\, 10^{-(n-1)} + a_n \cdot 10^{-n}$ with a_n such that $p_{n+1}^2 < 2$ while $(1.41...a_{n-1}(a_n + 1))^2 > 2$. □

1.1. Pythagoras will try to argue with certain triangles, but we modern people do not need the help of triangles to understand the meaning of equations... Furthermore, what about equations like $x^4 - 1 = 0$? It definitely looks more like $x^2 - 1 = 0$ than $x^2 + 1 = 0$ but would still require us to consider complex numbers.

1.2. Polynomial (with rational coefficient) = 0.

The above argument is problematic – in fact the statement of the lemma itself is already problematic. The problem is that we already assumed the existence of the irrational number $\sqrt{2}$. Even worse, we also assumed that this "number" enjoys all the addition/subtraction/multiplication, or even limit... properties of the good rational numbers!

But if we cannot assume a priori the existence of the limit, how do we know that the sequence should have a limit? This makes necessary the notion of "Cauchy sequences".

Definition 1.4. (Cauchy sequence) *A sequence of numbers $a_1, \ldots, a_n, \ldots$ (denoted $\{a_n\}$) is called a* Cauchy sequence *if the following holds:*

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall m, n > N, \qquad |a_m - a_n| < \varepsilon. \tag{1.1}$$

Think:

- Why is $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that $\forall n > N$, $|a_{n+1} - a_n| < \varepsilon$ not enough?

Example 1.5. Show that

- $\{a_n\}$, $a_n = 3$ for all n is a Cauchy sequence.
- $\{a_n\}$, $a_n = 1.4142...$ as defined in Lemma 1.3 is a Cauchy sequence.

Proof. We give detailed proof of the second claim here. By definition, we need to show that, given any $\varepsilon > 0$, we can find $N > 0$, such that whenever $n, m > N$, $|a_n - a_m| < \varepsilon$. The key is to find N based on the size of ε.

Given any $\varepsilon > 0$. We can always find a number M such that $\varepsilon > 10^{-M}$. Now take $N = M + 1$. For any $n, m > N$, we know from the construction of the sequence that a_n, a_m share the first N digits, in other words

$$|a_n - a_m| \leqslant 10^{-(N-1)} = 10^{-M} < \varepsilon. \tag{1.2}$$

Thus the proof ends. □

Example 1.6. Show that $a_n = \frac{1}{n} - \frac{1}{n+1}$ is Cauchy.

Proof. Given any $\varepsilon > 0$. Take N such that $N^2 > 2/\varepsilon$. Then for every $n, m > N$, we have

$$|a_n - a_m| \leqslant |a_n| + |a_m| = \frac{1}{n\,(n+1)} + \frac{1}{m\,(m+1)} < \frac{2}{N^2} < \varepsilon. \tag{1.3}$$

The proof ends. □

So we have shown that $a_n = 1.414...$ form a Cauchy sequence. To see that it cannot have a limit $a \in \mathbb{Q}$, we prove by contradiction, showing that if the limit is in $\mathbb{Q}$, then its square must be 2.

By construction of a_n, we know that $a_n^2 < 2$ while $\left(a_n + 10^{-(n-1)}\right)^2 > 2$. This means

$$|2 - a_n^2| = 2 - a_n^2 < \left(a_n + 10^{-(n-1)}\right)^2 - a_n^2 = 2 \cdot 10^{-(n-1)}\, a_n + 10^{-2(n-1)} < 10^{-(n-2)}. \tag{1.4}$$

Here the last step follows from the observation that $a_n^2 < 2$ implies $a_n < 2$. Thus we have $a_n^2 \longrightarrow 2$. On the other hand, by assumption $a_n \longrightarrow a$, we have

$$|a^2 - a_n^2| = |a - a_n|\, |a + a_n| < 4\, |a - a_n| \rightarrow 0. \tag{1.5}$$

Putting everything together we have $a^2 = 2$. Contradiction.

1.1.2. Construction of $\mathbb{R}$.

Naïvely, we want to simply consider the set of all Cauchy sequences, and call this set $\mathbb{R}$.

However there is a problem here. It is easy to realize that seemingly different Cauchy sequences may "converge" to the same number. For example consider $\{x_n = 1/n\}$ and $\{y_n = 0\}$. To fix this, we introduce the following (Of course, the tricky issue here is to tell that two Cauchy sequences converge to the same number **without** acknowledging the existence of this number!)

Definition 1.7. (Equivalence of Cauchy sequences) *Two Cauchy sequences $\{a_n\}, \{b_n\}$ are equivalent if their "mix-up" $\{c_n\}$, defined as*

$$c_n = \begin{cases} a_k & n = 2k \\ b_k & n = 2k-1 \end{cases} \tag{1.6}$$

is still a Cauchy sequence. We denote this relation as $\{a_n\} \sim \{b_n\}$.

The following lemma makes proving equivalence easier.

Lemma 1.8. *$\{a_n\} \sim \{b_n\}$ if and only if*

$$\forall \varepsilon\ \exists N \in \mathbb{N} \text{ such that } \forall n > N, \qquad |a_n - b_n| < \varepsilon. \tag{1.7}$$

Proof. Recall that "if and only if" means $\Longleftrightarrow$, which means

$$\{a_n\} \sim \{b_n\} \Longrightarrow \forall \varepsilon\ \exists N \in \mathbb{N} \text{ such that } \forall n > N, \qquad |a_n - b_n| < \varepsilon. \tag{1.8}$$

and

$$\forall \varepsilon\ \exists N > 0 \text{ such that } \forall n \in \mathbb{N}, \qquad |a_n - b_n| < \varepsilon \Longrightarrow \{a_n\} \sim \{b_n\}. \tag{1.9}$$

Let $\{c_n\}$ be defined as

$$c_n = \begin{cases} a_k & n = 2k \\ b_k & n = 2k-1 \end{cases} \tag{1.10}$$

- $\Longrightarrow$. Consider any $\varepsilon > 0$. Since $\{a_n\} \sim \{b_n\}$, there is $N_1 \in \mathbb{N}$ such that $|c_n - c_m| < \varepsilon$ for any $n, m > N_1$. Taking $N \in \mathbb{N}$ satisfying $N > N_1/2+1$. Then for any $n > N$, we have $a_n = c_{2n}, b_n = c_{2n-1}$ therefore

$$|a_n - b_n| = |c_{2n} - c_{2n-1}| < \varepsilon \tag{1.11}$$

  as $2n, 2n-1 > N_1$.

- $\Longleftarrow$. Consider any $\varepsilon > 0$. Let $N_1 \in \mathbb{N}$ be such that $\forall n > N_1$, $|a_n - b_n| < \varepsilon$. Let $N_2 \in \mathbb{N}$ be such that $\forall n, m > N_2$, $|a_n - a_m| < \varepsilon$. Let $N_3 \in \mathbb{N}$ be such that $\forall n, m > N_3$, $|b_n - b_m| < \varepsilon$. Note that such N_2, N_3 exists because $\{a_n\}, \{b_n\}$ are Cauchy sequences.

  Now let $N = 2\max\{N_1, N_2, N_3\} + 1$.

  For any $n, m > N$, we have

$$|c_n - c_m| = \begin{cases} |a_k - a_l| & n = 2k, m = 2l \\ |b_k - b_l| & n = 2k+1, m = 2l+1 \\ |a_k - b_l| & n = 2k, m = 2l+1 \\ |b_k - a_l| & n = 2k+1, m = 2l \end{cases}. \tag{1.12}$$

  As $N = 2\max\{N_1, N_2, N_3\} + 1$, $k, l > \max\{N_1, N_2, N_3\}$ in every case. Therefore $|c_n - c_m| < \varepsilon$. The proof ends. □

Lemma 1.9. ($\sim$ is indeed an equivalence relation) *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be Cauchy sequences. We have*

- *$\{a_n\} \sim \{a_n\}$;*

- *If* $\{a_n\} \sim \{b_n\}$ *then* $\{b_n\} \sim \{a_n\}$*;*
- *If* $\{a_n\} \sim \{b_n\}$ *and* $\{b_n\} \sim \{c_n\}$*, then* $\{a_n\} \sim \{c_n\}$*.*

Proof. We only prove the last claim here. The first two are trivial.

Thanks to Lemma 1.8, to show that $\{a_n\} \sim \{c_n\}$ we only need to show for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - c_n| < \varepsilon$.

Consider any $\varepsilon > 0$. Since $\{a_n\} \sim \{b_n\}$ and $\{b_n\} \sim \{c_n\}$, there is $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $|a_n - b_n| < \varepsilon/2$, and there is $N_2 \in \mathbb{N}$ such that for all $n > N_2$, $|b_n - c_n| < \varepsilon/2$. Now take $N = \max\{N_1, N_2\}$. We have for any $n > N$,

$$|a_n - c_n| \leqslant |a_n - b_n| + |b_n - c_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \tag{1.13}$$

The proof ends. □

Now we are ready to define $\mathbb{R}$: We put all equivalent Cauchy sequences together and treat it as one object – an equivalence class of Cauchy sequences, and call the collection of such equivalence classes $\mathbb{R}$.

Definition 1.10. (Real numbers) *The real numbers* $\mathbb{R}$ *is the set of all equivalence classes of Cauchy sequences.*

Example 1.11. For example, any rational number $\{q\}$ is the equivalence class of Cauchy sequences equivalent to the particular Cauchy sequence $\{a_n = q\}$. $\sqrt{2}$ is the class of all Cauchy sequences equivalent to $\{1, 1.4, 1.41, \ldots\}$.

1.1.3. Is this $\mathbb{R}$ the $\mathbb{R}$ we want?.

Put another way, can we naturally do addition, subtraction, multiplication, division, and taking limit in this newly constructed set $\mathbb{R}$? The answer is yes but the justification is too technical/long to be included here. Instead we simply state the results without proofs.

Lemma 1.12. (Arithmetics in $\mathbb{R}$ are well-defined) *Let* $\{a_n\}, \{b_n\}$ *be Cauchy sequences. Then so are* $\{a_n \pm b_n\}$*,* $\{a_n\, b_n\}$*, and* $\{a_n / b_n\}$ *if* $\{b_n\}$ *does not converge to* 0*. Furthermore, if* $\{\tilde{a}_n\}$*,* $\{\tilde{b}_n\}$ *are another two sequences that are equivalent to* $\{a_n\}, \{b_n\}$*, that is* $\{a_n\} \sim \{\tilde{a}_n\}$*,* $\{b_n\} \sim \{\tilde{b}_n\}$*, then*

$$\{a_n \pm b_n\} \sim \{\tilde{a}_n \pm \tilde{b}_n\}; \qquad \{a_n\, b_n\} \sim \{\tilde{a}_n\, \tilde{b}_n\}; \qquad \{a_n / b_n\} \sim \{\tilde{a}_n / \tilde{b}_n\}. \tag{1.14}$$

Theorem 1.13. $\mathbb{R}$ *constructed as in the last section satisfies:*

1. *it is an ordered field;*[1.3]
2. *It has the "least upper bound" property: For any subset* $E \subseteq \mathbb{R}$*, if there is an upper bound, that is* $a \in \mathbb{R}$ *such that* $a \geqslant x$ *for every* $x \in E$*, then there is a "least upper bound"* $a_{\min} \in \mathbb{R}$*. In other words, the set* $A = \{a \in \mathbb{R} : a \text{ is an upper bound for } E\}$ *has a minimum element.*

We will see soon that the "least upper bound" property guarantees that a Cauchy sequence in $\mathbb{R}$ converges and indeed has a limit again in $\mathbb{R}$.[1.4]

1.3. *We do not go into details here, it suffices to say that one can naturally do all the arithmetics and comparison* ($<, >$) *in an ordererd field.*

1.4. That is $\mathbb{R}$ is "complete".

1.2. Limits.

From now on we take the existence of $\mathbb{R}$ and all its properties for granted. In particular, we assume that the "least upper bound" property holds.

1.2.1. Definition.

Definition 1.14. *A sequence of real numbers* $\{x_n\}$ *is said to* converge *to a real number* $a \in \mathbb{R}$ *if and only if*

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N, \qquad |x_n - a| < \varepsilon. \tag{1.15}$$

We denote this convergence by $x_n \longrightarrow a$*, or* $\lim_{n \longrightarrow \infty} x_n = a$*.*

Think: Suppose we have proved

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > 5\,N, \qquad |x_n - a| < 20\,\varepsilon, \tag{1.16}$$

can we conclude $x_n \longrightarrow a$? What if we proved

$$\forall m \in \mathbb{N}\ \exists N \in \mathbb{N} \text{ such that } \forall n > N^2, \qquad |x_n - a| < \frac{1}{\sqrt{m}}, \tag{1.17}$$

can we conclude $x_n \longrightarrow a$?

Remark 1.15. Definitions and theorems in analysis are unique in the sense that you can make many harmless changes. However it is not possible to tell whether a change is harmless or not without enough understanding.

Example 1.16. Show that $\left\{x_n = \sum_{k=1}^n \frac{1}{k\,(k+1)}\right\}$ converges to 1 as $n \nearrow \infty$.

Proof. We need to show that for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$, $\left|\sum_{k=1}^n \frac{1}{k\,(k+1)} - 1\right| < \varepsilon$. To do this we first try to simplify

$$\sum_{k=1}^n \frac{1}{k\,(k+1)} = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}. \tag{1.18}$$

Now we know how to choose N. Given any $\varepsilon > 0$, choose N such that $\varepsilon > 1/N$. Then for all $n > N$, we have

$$\left|\sum_{k=1}^n \frac{1}{k\,(k+1)} - 1\right| = \frac{1}{n+1} < \frac{1}{N} < \varepsilon. \tag{1.19}$$

Thus ends the proof. □

Example 1.17. (Nonexistence of limit) Show that $\{(-1)^n\}$ has no limit.

Proof. (Proof by contradiction) Assume the contrary, that is $(-1)^n \longrightarrow a$ for some a. Then for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$, $|(-1)^n - a| < \varepsilon$.

We take $\varepsilon = 1$. There is $N \in \mathbb{N}$ such that for all $n > N$, $|(-1)^n - a| < 1$. Since this holds for all $n > N$, in particular it holds for all odd $n > N$, thus $|-1 - a| < 1$. On the other hand it also holds for all even $n > N$, which leads to $|1 - a| < 1$. Combining the two together we get

$$|-1 - a| + |1 - a| < 2. \tag{1.20}$$

But on the other hand we have

$$|-1 - a| + |1 - a| = |1 + a| + |1 - a| \geqslant |1 + a + 1 - a| = |2| = 2. \tag{1.21}$$

Therefore $2 < 2$. Contradiction! □

Remark 1.18. Note that if we do not use proof by contradiction, things may get a bit complicated, as we need to show that

$$\forall a \in \mathbb{R}\ \exists \varepsilon_0 > 0\ \forall N \in \mathbb{N}\ \exists n > N, \qquad |(-1)^n - a| \geqslant \varepsilon_0. \tag{1.22}$$

Example 1.19. Show that $\{\log n\}$ has no limit.

Proof. Again we prove by contradiction. Assume the contrary: that is $\log n \longrightarrow a$ for some $a \in \mathbb{R}$. Then there is $N \in \mathbb{N}$ such that for all $n > N$, $|\log n - a| < 1$. In particular, for all $n > N$ we must have $\log n < |a| + 1$. Now take $N' \in \mathbb{N}$ such that $N' > \exp(|a| + 1)$ and take an $n_0 \in \mathbb{N}$ such that $n_0 > \max\{N, N'\}$. Then we have $\log n_0 > \log N' > |a| + 1$. Contradiction. □

1.2.2. Properties of limits.

Lemma 1.20. *A sequence can have at most one limit.*

Proof. We prove by contradiction. Assume $x_n \longrightarrow a$ and $x_n \longrightarrow b$ with $a \neq b$. Without loss of generality we assume $b > a$.

Take $\varepsilon = \frac{b-a}{2}$. There is $N_1 \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n > N_1$; There is $N_2 \in \mathbb{N}$ such that $|x_n - b| < \varepsilon$. Now take $N = \max\{N_1, N_2\}$. Now take any $n > N$, we have $n > N_1, n > N_2$ and therefore

$$|x_n - a| < \varepsilon, |x_n - b| < \varepsilon \tag{1.23}$$

which combined to give

$$b - a = |b - a| \leqslant |b - x_n| + |x_n - a| = |x_n - b| + |x_n - a| < 2\,\varepsilon = b - a. \tag{1.24}$$

Contradiction! □

Proposition 1.21. (Manipulation of limits) *Let $x_n \longrightarrow a$, $y_n \longrightarrow b$. Then*

a) $x_n \pm y_n \longrightarrow a \pm b$;

b) $x_n\, y_n \longrightarrow a\, b$;

c) *If $b \neq 0$, then $x_n / y_n \longrightarrow a/b$.*

Proof. We only give detailed proof for b). Other cases are left as exercises.

To show $x_n\, y_n \longrightarrow a\, b$, all we need is for any given $\varepsilon > 0$, find $N \in \mathbb{N}$ such that for all $n > N$, $|x_n\, y_n - a\, b| < \varepsilon$. First assume that $a, b \neq 0$.

Let $N_1 \in \mathbb{N}$ be such that $|y_n - b| < |b|$. Now for any given $\varepsilon > 0$, let N_2 be such that $|x_n - a| < \frac{\varepsilon}{4\,|b|}$, and N_3 be such that $|y_n - b| < \frac{\varepsilon}{2\,|a|}$.

Take $N = \max\{N_1, N_2, N_3\}$. For any $n > N$, we have

$$\begin{aligned} |x_n\, y_n - a\, b| &= |(x_n - a)\, y_n + a\,(y_n - b)| \\ &\leqslant |(x_n - a)|\,|y_n| + |a|\,|y_n - b| \\ &< \frac{\varepsilon}{4\,|b|}\, 2\,|b| + |a|\,\frac{\varepsilon}{2\,|a|} \\ &= \varepsilon. \end{aligned}$$

When either a or b is (or both are) 0, all we need to show is $x_n\, y_n \longrightarrow 0$. The method is the same and the argument is simpler so omitted. □

Theorem 1.22. (Comparison of limits) *Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences. If there is $N_0 \in \mathbb{N}$ such that $x_n \leqslant y_n$ for all $n \geqslant N_0$, then*

$$\lim_{n \longrightarrow \infty} x_n \leqslant \lim_{n \longrightarrow \infty} y_n. \tag{1.25}$$

Proof. Let $x \longrightarrow a$, $y — b$. We prove by contradiction. Assume $a > b$. Set $\varepsilon = \frac{a-b}{2}$. Then there is $N_1 \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n > N_1$, while $N_2 \in \mathbb{N}$ such that $|y_n - b| < \varepsilon$ for all $n > N_2$. Take $n > \max\{N_1, N_2\}$. Then we have

$$x_n \geqslant a - |x_n - a| > a - \varepsilon = \frac{a+b}{2}; \tag{1.26}$$

$$y_n \leqslant b + |y_n - b| < b + \varepsilon = \frac{a+b}{2}. \tag{1.27}$$

But this combined with $x_n \leqslant y_n$ gives $\frac{a+b}{2} < \frac{a+b}{2}$, contradiction. $\square$

Example 1.23. (Important!) Show by an counterexample that the following is false:

> Suppose $\{x_n\}, \{y_n\}$ are convergent sequences. If there is $N_0 \in \mathbb{N}$ such that $x_n < y_n$ for all $n \geqslant N_0$, then $\lim_{n \longrightarrow \infty} x_n < \lim_{n \longrightarrow \infty} y_n$.

The simplest example is $x_n = 1/n$, $y_n = -1/n$.

Theorem 1.24. (Squeeze Theorem) *Let $x_n \rightarrow a$ and $y_n \longrightarrow a$. Let $\{w_n\}$ be a sequence. Assume that there is $N_0 \in \mathbb{N}$ such that for all $n \geqslant N_0$,*

$$x_n \leqslant w_n \leqslant y_n, \tag{1.28}$$

then

a) w_n converges.

b) $w_n \longrightarrow a$.

Remark 1.25. Of course proving $w_n \longrightarrow a$ suffices. However we would like to emphasize that, the significance of this theorem is that the relation (1.28) "forces" w_n to converge.

Proof. For any given $\varepsilon > 0$, let $N_1 \in \mathbb{N}$ be such that $|x_n - a| < \varepsilon$ for all $n > N_1$; Let $N_2 \in \mathbb{N}$ be such that $|y_n - a| < \varepsilon$ for all $n > N_2$. Now let $N = \max\{N_1, N_2\}$. For all $n > N$, we have

$$|w_n - a| \leqslant \max\{|x_n - a|, |y_n - a|\} < \varepsilon. \tag{1.29}$$

Thus ends the proof. $\square$

Corollary 1.26. *Let $\{w_n\}$ be a sequence. If there is another sequence $x_n \longrightarrow 0$ such that $|w_n| \leqslant x_n$, then $w_n \longrightarrow 0$. In particular, if $|w_n| \longrightarrow 0$, then $w_n \longrightarrow 0$.*

The proof is left as exercise.

Example 1.27. Find $\lim_{n \longrightarrow \infty} 2^{-n} \sin(n^8)$.

We simply apply the above corollary: $2^{-n} \longrightarrow 0$,

$$|2^{-n} \sin(n^8)| \leqslant 2^{-n} \tag{1.30}$$

therefore the limit is also 0.

1.2.3. Relation to Cauchy sequences.

In general, to prove convergence for $\{x_n\}$ with formulas for x_n given, we need to do two things:

1. Guess the limit a.
2. Prove that indeed $x_n \longrightarrow a$ using either definition or properties (see Section 1.2.2 below) or both.

On the other hand, there are many situations where we do not have explicit formulas for x_n and cannot guess what the limit is. The following is the most useful result in those situations.

Theorem 1.28. (Relation to Cauchy sequences) *Let $\{x_n\}$ be a sequence. Then $\{x_n\}$ converges to some real number $a \in \mathbb{R}$ if and only if it is a Cauchy sequence.*

Proof. It's "if and only if", so we need to show the "only if":

$$x_n \longrightarrow a \text{ for some } a \in \mathbb{R} \Longrightarrow \{x_n\} \text{ is Cauchy} \tag{1.31}$$

and the “if”:

$$\{x_n\} \text{ is Cauchy} \Longrightarrow x_n \longrightarrow a \text{ for some } a \in \mathbb{R}. \tag{1.32}$$

- ”Only if”. Given any $\varepsilon > 0$, we need to find $N \in \mathbb{N}$ such that for all $n, m > N$, $|x_n - x_m| < \varepsilon$. We proceed as follows. For the given $\varepsilon > 0$, since $x_n \longrightarrow a$, there is $N \in \mathbb{N}$ such that

  $$|x_n - a| < \varepsilon/2 \tag{1.33}$$

  for all $n > N$. Now for any $n, m > N$, we have

  $$|x_n - a| < \varepsilon/2, \qquad |x_m - a| < \varepsilon/2 \tag{1.34}$$

  which combined to give

  $$|x_n - x_m| = |(x_n - a) + (a - x_m)| \leqslant |x_n - a| + |a - x_m| = |x_n - a| + |x_m - a| < \varepsilon. \tag{1.35}$$

- ”If”. For this part we need to use the least upper bound property of $\mathbb{R}$. Intuitively, $\{x_n\}$ is Cauchy means they cluster around something, but to show that “something” is a real number, we need the property that $\mathbb{R}$ has no “holes”.

  Let the set $A = \{a \in \mathbb{R}: \exists N \in \mathbb{N}, \text{ such that } a < x_n \text{ for all } n > N\}$. We first show that A has an upper bound. Since $\{x_n\}$ is Cauchy, there is $N_1 \in \mathbb{N}$ such that $|x_n - x_m| < 1$ for all $n, m > N_1$. In particular we have

  $$x_n < x_{N_1+1} + 1 \tag{1.36}$$

  for all $n > N_1$. Now take

  $$b = \max\{x_1 + 1, x_2 + 1, \ldots, x_{N_1} + 1, x_{N_1+1} + 1\} \tag{1.37}$$

  we have $b > x_n$ for all $n \in \mathbb{N}$ and clearly $b > a$ for every $a \in A$. Using the same method one can show that A is not empty.

  Thanks to the least upper bound property, we have $b_{\min} \in \mathbb{R}$ such that $b_{\min} \geqslant a$ for every $a \in A$ while for every $b < b_{\min}$ there is $a \in A$ with $a > b$. We prove the $x_n \longrightarrow b_{\min}$.

  Given any $\varepsilon > 0$. We know that there is $a \in A$ such that $a > b_{\min} - \varepsilon$. Therefore there is $N_1 \in \mathbb{N}$ such that $b_{\min} - \varepsilon < x_n$ for all $n > N_1$;

  Now we show the existence of $N_2 \in \mathbb{N}$ such that $b_{\min} + \varepsilon > x_n$ for all $n > N_2$. We prove by contradiction. Assume that for every $N \in \mathbb{N}$, there is $n > N$ such that $x_n \geqslant b_{\min} + \varepsilon$. Since $\{x_n\}$ is a Cauchy sequence, there is $N_3 \in \mathbb{N}$ such that for all $n, m > N_3$, $|x_n - x_m| < \varepsilon/2$. For this N_3 we can find $l > N_3$ with $x_l \geqslant b_{\min} + \varepsilon$. Consequently, for all $n > N_3$, we have

  $$x_n \geqslant x_l - |x_n - x_l| > b_{\min} + \varepsilon/2. \tag{1.38}$$

  But this means $b_{\min} + \varepsilon/2 \in A$, contradicting $b_{\min} \geqslant a$ for every $a \in A$.

  Finally take $N = \max\{N_1, N_2\}$. For every $n > N$, we have

  $$b_{\min} - \varepsilon < x_n < b_{\min} + \varepsilon \Longrightarrow |x_n - b_{\min}| < \varepsilon. \tag{1.39}$$

  Therefore $x_n \longrightarrow b_{\min}$ and the proof for “if” ends. □

The above theorem is most useful when the sequence is given iteratively.

Example 1.29. Let $x_0 > 2$. Define x_n iteratively by

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2\,x_n}. \tag{1.40}$$

Prove that $x_n \longrightarrow \sqrt{2}$.

Proof. The idea is to show that it is Cauchy. Once this is done, we can take limits of both sides to reach (denote the limit by a):

$$a = a - \frac{a^2 - 2}{2\,a} \Longrightarrow a^2 = 2 \Longrightarrow a = \pm\sqrt{2}. \tag{1.41}$$

From this we see that we need to prove the following:

- First show that $x_n > 0$. We use mathematical induction. Let $P(n)$ denote the statement "$x_n > 0$". Mathematical induction consists of two steps:

  1. $P(0)$ is true.[1.5] We have $x_0 > 2 > 0$.
  2. If $P(m)$ is true then $P(m+1)$ is true. We have

  $$x_{m+1} = x_m - \frac{x_m^2 - 2}{2\,x_m} = \frac{x_m^2 + 2}{2\,x_m}. \tag{1.42}$$

  Therefore if $x_m > 0$, $x_{m+1} > 0$.

- Since a ratio is involved, we need to show $x_n \neq 0$ and $a \neq 0$. We do this through showing $x_n^2 \geqslant 2$ for all $n \in \mathbb{N}$. This can be done directly as follows:

$$x_n^2 - 2 = \left(x_{n-1} - \frac{x_{n-1}^2 - 2}{2\,x_{n-1}} \right)^2 - 2 = \left(\frac{x_{n-1}^2 + 2}{2\,x_{n-1}} \right)^2 - 2 = \left(\frac{x_{n-1}^2 - 2}{2\,x_{n-1}} \right)^2 \geqslant 0. \tag{1.43}$$

- x_n is Cauchy. The basic idea is to show that $|x_n - x_{n+1}| \leqslant M\,r^n$ for some $0 < r < 1$ and some $M > 0$ (Why this is enough is left as exercise). Taking difference of

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2\,x_n} = \frac{x_n}{2} + \frac{1}{x_n} \text{ and } x_n = x_{n-1} - \frac{x_{n-1}^2 - 2}{2\,x_{n-1}} = \frac{x_{n-1}}{2} + \frac{1}{x_{n-1}} \tag{1.44}$$

  we have

$$x_{n+1} - x_n = \left[\frac{1}{2} - \frac{1}{x_n\,x_{n-1}} \right] (x_n - x_{n-1}). \tag{1.45}$$

  Therefore

$$|x_{n+1} - x_n| = \left| \frac{1}{2} - \frac{1}{x_n\,x_{n-1}} \right| |x_n - x_{n-1}|. \tag{1.46}$$

  As $x_n > \sqrt{2}$ we have $x_n\,x_{n-1} \geqslant 2$ so $\left| \frac{1}{2} - \frac{1}{x_n\,x_{n-1}} \right| \leqslant \frac{1}{2}$. This leads to

$$|x_{n+1} - x_n| \leqslant \frac{1}{2} |x_n - x_{n-1}| \leqslant \frac{1}{2^2} |x_{n-1} - x_{n-2}| \leqslant \cdots \leqslant \left(\frac{1}{2} \right)^n |x_1 - x_0|. \tag{1.47}$$

Since x_n is Cauchy, there is $a \in \mathbb{R}$ such that $x_n \longrightarrow a$. By (1.41) we have $a = \pm\sqrt{2}$. Since $x_n > 0$, we conclude $a = \sqrt{2}$ (using Comparison Theorem 1.22). ☐

Remark 1.30. The above is a very effective way to compute $\sqrt{2}$.

1.2.4. Understanding convergence.

In this section we understand what convergence means. Or more precisely, we understand what happens when a sequence is **not** convergent. Turns out, there are only two situations where a sequence does not converge:

1. The sequence is unbounded.
2. The sequence is oscillating and the amplitude does not tend to 0.

Definition 1.31. (Boundedness) *A sequence* $\{x_n\}$ *is said to be*

- *bounded above if there is a number* $M \in \mathbb{R}$ *such that* $x_n \leqslant M$ *for all* n*,*
- *bounded below if there is a number* $M \in \mathbb{R}$ *such that* $x_n \geqslant M$ *for all* n*,*
- *bounded if there is a number* $M \in \mathbb{R}$ *such that* $|x_n| \leqslant M$ *for all* n*.*

1.5. Because we start from x_0.

Lemma 1.32. (Relation between boundedness) *$\{x_n\}$ is bounded if and only if it is bounded above and below.*

Proof.

- "If". If $\{x_n\}$ is bounded above and below, then there are $M_1, M_2 \in \mathbb{R}$ such that $M_1 \leqslant x_n \leqslant M_2$ for all $n \in \mathbb{N}$. Now take $M = \max\{|M_1|, |M_2|\}$. We have

$$-M \leqslant -|M_1| \leqslant M_1 \leqslant x_n \leqslant M_2 \leqslant |M_2| \leqslant M \tag{1.48}$$

  for all $n \in \mathbb{N}$.

- "Only if". If $\{x_n\}$ is bounded, then there is $M \in \mathbb{R}$ such that $-M \leqslant x_n \leqslant M$. Thus $\{x_n\}$ is bounded both above and below. □

Lemma 1.33. *If $x_n \longrightarrow a$, then x_n is bounded. That is every convergent sequence is bounded.*

Proof. Take $\varepsilon = 1$. Since $x_n \longrightarrow a$, there is $N \in \mathbb{N}$ such that for all $n > N$, we have $|x_n - a| < \varepsilon$. Thus for all $n > N$, we have

$$|x_n| \leqslant |a| + |x_n - a| < |a| + 1. \tag{1.49}$$

Now set

$$M = \max\{|x_1|, \ldots, |x_N|, |a| + 1\}. \tag{1.50}$$

We have $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. Thus ends the proof. □

Note. Spot the mistake in the following proof:
Take $\varepsilon = |a|$. Since $x_n \longrightarrow a$, ... (the remaining the same as that in the above proof)

Definition 1.34. (Subsequence) *A* subsequence *of a sequence $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of the form $\{x_{n_k}\}_{k\in\mathbb{N}}$, where each $n_k \in \mathbb{N}$ with $n_1 < n_2 < n_3 < \cdots$.*

It is clear that a subsequence of a subsequence is a subsequence of the original sequence.

Lemma 1.35. *If $x_n \longrightarrow a$, then every of its subsequences also converge to a.*

Proof. Let $\{x_{n_k}\}$ be one subsequence. Given any $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that for all $n > N$, $|x_n - a| < \varepsilon$. Now let $K \in \mathbb{N}$ be such that for all $k > K$, $n_k > N$. With such K we see that for all $k > K$, we have

$$|x_{n_k} - a| < \varepsilon. \tag{1.51}$$

Thus ends the proof. □

On the other hand, we have the following:

Lemma 1.36. *Let $\{x_n\}$ be a sequence. If all its subsequences converge to the same $a \in \mathbb{R}$, then $x_n \longrightarrow a$.*

Proof. Assume $\{x_n\}$ does not converge to a. All we need to do is to find one subsequence which does not converge to a[1.6]. As $\{x_n\}$ does not converge to a, we have

$$\exists \varepsilon_0 > 0 \ \forall N \in \mathbb{N} \ \exists n > N, \qquad |x_n - a| > \varepsilon_0. \tag{1.52}$$

Now first take $N = 1$. There is $n_1 > 1$ such that $|x_{n_1} - a| > \varepsilon_0$. Next take $N = n_1$. There is $n_2 > N = n_1$ such that $|x_{n_2} - a| > \varepsilon_0$. Next take $N = n_2$ and do the same and so on. This way we obtain a subsequence $\{x_{n_k}\}$ satisfying $|x_{n_k} - a| > \varepsilon_0$ for all $k \in \mathbb{N}$. This subsequence does not converge to a. □

We need one more result to fully understand convergence of sequences.

1.6. Whether it converges to something else or does not converge at all does not matter here.

1.3. Bolzano-Weierstrass.

1.3.1. The theorem and its consequences.

We first state the theorem and discuss its significance. The proof is postponed to Section 1.3.3.

Theorem 1.37. **(Bolzano-Weierstrass)** *Let $\{x_n\}$ be bounded a sequence of real numbers. Then there is a converging subsequence.*

Remark 1.38. The theorem not only tell us something about simple cases, such as $(-1)^n$ for which we can easily get a convergent subsequence, it also tell us sequences like $\{\sin n\}$ also has convergent subsequence(s).[1.7]

Theorem 1.39. **(Classification of sequences)** *Let $\{x_n\}$ be a sequence of real numbers. Then one of the following is true:*

a) x_n converges;

b) $\{x_n\}$ is not bounded.

c) There are two convergent subsequences with different limits.

Remark 1.40. a) and b),c) are clearly mutually exclusive. On the other hand both b) c) may be true for a single sequence. For example let $a_n = \begin{cases} 1 & n = 4k+1 \\ n & n = 4k+2 \\ -1 & n = 4k+3 \\ -n & n = 4k+4 \end{cases}$ for $k \in \mathbb{N}$.

Proof. It is clear that a) and b) are mutually exclusive. The only thing left to show is that if $\{x_n\}$ is bounded and does not converge, then it has two converging subsequences with different limits.

Since $\{x_n\}$ is bounded, by the Bolzano-Weierstrass theorem there is a converging subsequence. We denote its limit by a. Since $x_n \not\longrightarrow a$, by definition there must be $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$ there is $n > N$ with $|x_n - a| > \varepsilon_0$. From this we can get a subsequence satisfying $|x_{n_k} - a| > \varepsilon_0$ as follows:

i. Take $N = 1$. There is $|x_{n_1} - a| > \varepsilon_0$;

ii. Take $N = n_1$. Find $|x_{n_2} - a| > \varepsilon_0$;

iii. Take $N = n_2$, ...

Now apply the Bolzano-Weierstrass theorem to this subsequence. We see that there is a converging subsequence (to this subsequence), called $\{x_{n_{k_l}}\}$ $(l = 1, 2, 3, \ldots)$. Let $x_{n_{k_l}} \longrightarrow a'$ as $l \longrightarrow \infty$. We show that $a' \neq a$. Assume the contrary. Then $x_{n_{k_l}} \longrightarrow a$. By definition there is $N \in \mathbb{N}$ such that $\left|x_{n_{k_l}} - a\right| < \varepsilon_0$ for all $l > N$. But this contradicts the fact that $x_{n_{k_l}}$ are chosen from x_{n_k} which satisfies $|x_{n_k} - a| > \varepsilon_0$ for all $k \in \mathbb{N}$. $\square$

1.3.2. Monotone sequences.

To prove the Bolzano-Weierstrass theorem, we first study monotone sequences.

Definition 1.41. *A sequence $\{x_n\}$ is increasing if $x_{n+1} \geqslant x_n$ for all n (strictly increasing of $x_{n+1} > x_n$ for all $n \in \mathbb{N}$); A sequence is decreasing if $x_{n+1} \leqslant x_n$ for all n (strictly decreasing if $x_{n+1} < x_n$ for all $n \in \mathbb{N}$); A sequence is monotone if it is either increasing or decreasing.*

Convergence results when monotonicity meets boundedness.

1.7. In fact, for every $-1 \leqslant a \leqslant 1$, there is a subsequence of $\{\sin n\}$ convergent to a.

Lemma 1.42. *Let $\{x_n\}$ be monotone increasing (decreasing) and bounded above (below). Then $x_n \longrightarrow a$ for some $a \in \mathbb{R}$.*

Proof. Since $\{x_n\}$ is bounded above. There is $M \in \mathbb{R}$ such that $x_n \leqslant M$ for all $n \in \mathbb{N}$.

We now show that x_n is Cauchy through proof by contradiction. Assume x_n is not Cauchy, then there is $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$, there are $n, m > N$ with $|x_n - x_m| > \varepsilon_0$.

Now take $N = 1$. We find $n_2 > n_1 > 1$ with $|x_{n_2} - x_{n_1}| > \varepsilon_0$. Next take $N = n_2$, we have $n_4 > n_3 > n_2$ with $|x_{n_4} - x_{n_3}| > \varepsilon_0$. Doing this repeatedly, we obtain a (still increasing) subsequence x_{n_k} such that $|x_{n_{2k}} - x_{n_{2k-1}}| > \varepsilon_0$. Since this subsequence is still increasing,

$$x_{n_{2k}} \geqslant (x_{n_{2k}} - x_{n_{2k-1}}) + (x_{n_{2k-2}} - x_{n_{2k-3}}) + \cdots + (x_{n_2} - x_{n_1}) + x_{n_1} > k\,\varepsilon_0 + x_{n_1}. \tag{1.53}$$

Taking $k > (M - x_{n_1})/\varepsilon_0$ we reach $x_{n_{2k}} > M$. Contradiction. □

Remark 1.43. An alternative proof is as follows: Let $A = \{a \in \mathbb{R}: a < x_n \text{ for some } n \in \mathbb{N}\}$. We can show that A has an upper bound and is nonempty. Applying the least upper bound property of $\mathbb{R}$, there is $b_{\min} \in \mathbb{R}$ such that $b_{\min} \geqslant a$ for all $a \in A$, and for every $b < b_{\min}$, there is $a \in A$ satisfying $a > b$. Then prove $x_n \longrightarrow b_{\min}$. This approach is left as exercise.

1.3.3. Proof of Bolzano-Weierstrass.

All we need to do is to find a monotone subsequence from $\{x_n\}$.

Proof. (of Bolzano-Weierstrass) Consider the following subsequences of $\{x_n\}$: $x_m, x_{m+1}, x_{m+2}, \ldots$. There are only two cases:

- For every such sequence, there is a smallest element: $x_{n_1} = x_1$,

  $$x_{n_k} := \min \{x_{n_{k-1}+1}, x_{n_{k-1}+2}, \ldots\}. \tag{1.54}$$

  In this case we know that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and it is increasing and bounded above. Applying Lemma 1.42 to this subsequence we see that it converges to some $a \in \mathbb{R}$.

- There is one m, such that there is no smallest element in $\{x_{n_m+1}, x_{n_m+2}, \ldots\}$. Set $\tilde{x}_{n_1} = x_{n_m+1}$. Set n_2 to be the smallest number that $\tilde{x}_{n_2} < \tilde{x}_{n_1}$, that is $\tilde{x}_i \geqslant \tilde{x}_{n_1}$ for all $n_1 \leqslant i < n_2$. This can be done because there is no smallest number. Set n_3 to be the smallest number that $\tilde{x}_{n_3} < \tilde{x}_{n_2}$, and so on. This way we obtained a decreasing subsequence which is bounded below. Applying Lemma 1.42 to this subsequence we see that it converges to some $a \in \mathbb{R}$. □

1.4. Infinity Limits.

We have seen that a sequence has to be in exactly one of the following three situations:

1. It converges to a real number;
2. It oscillates with non-vanishing amplitude;
3. It is unbounded.

In this section we take a closer look at the third situation.

Example 1.44. Consider the following two sequences

$$x_n = n; \qquad y_n = n \sin(n). \tag{1.55}$$

Both are unbounded. However they are different: $\{x_n\}$ has a very predictable behavior as $n \longrightarrow \infty$ while $\{y_n\}$ does not.

Definition 1.45. *A sequence $\{x_n\}$ is said to diverge to $+\infty$, denoted $x_n \longrightarrow +\infty$, if for any $M \in \mathbb{R}$ there is $N \in \mathbb{N}$ such that for all $n > N$, $x_n > M$. Or in formal logic*

$$\forall M \in \mathbb{R}\ \exists N \in \mathbb{N} \text{ such that } \forall n > N, \qquad x_n > M. \tag{1.56}$$

Similarly, a sequence $\{x_n\}$ is said to diverge to $-\infty$, denoted $x_n \longrightarrow -\infty$, if

$$\forall M \in \mathbb{R}\ \exists N \in \mathbb{N} \text{ such that } \forall n > N, \qquad x_n < M. \tag{1.57}$$

Note. Often the $+$ in $+\infty$ is omitted.

Definition 1.46. $\mathbb{R}$ *together with $\pm\infty$ is called "extended real numbers". That is extended real numbers is the set* $\mathbb{R} \cup \{\infty, -\infty\}$.

Most results involving finite limits can be readily extended to include the cases $\rightarrow \pm\infty$, as long as these rules are followed:

$$x + \infty = \infty, \qquad x - \infty = -\infty, \qquad x \in \mathbb{R} \tag{1.58}$$
$$x \cdot \infty = \infty, \qquad x \cdot (-\infty) = -\infty, \qquad x > 0 \tag{1.59}$$
$$x \cdot \infty = -\infty, \qquad x \cdot (-\infty) = \infty, \qquad x < 0 \tag{1.60}$$
$$\infty + \infty = \infty, \qquad -\infty - \infty = -\infty \tag{1.61}$$
$$\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty \quad \infty \cdot (-\infty) = (-\infty) \cdot (\infty) = -\infty \tag{1.62}$$

and the following are not involved: $\infty - \infty$ or $0 \cdot (+\infty)$ or $0 \cdot (-\infty)$.

Proposition 1.47. (Extension of Proposition 1.21) *Let $x_n \longrightarrow a$, $y_n \longrightarrow b$ where a, b are extended real numbers. Then*

a) $x_n \pm y_n \longrightarrow a \pm b$;

b) $x_n y_n \longrightarrow a\, b$;

c) If $b \neq 0$, then $x_n / y_n \longrightarrow a / b$.

Proof. We only prove the product case and leave other cases as exercises.

When both $a, b \in \mathbb{R}$, the proof has been done in Proposition 1.21. Here we need to study the two new cases: $a, b = \pm\infty$ and $a \in \mathbb{R}, b = \pm\infty$ or $a = \pm\infty, b \in \mathbb{R}$.

- Case 1. We only prove for $a = b = \infty$. (The other three sub-cases can be proved similarly.) As $\infty \cdot \infty = \infty$, we need to show that for every $M \in \mathbb{R}$, there is $N \in \mathbb{N}$ such that for all $n > N$, $x_n y_n > M$.

  For every given $M \in \mathbb{R}$, since $x_n \longrightarrow \infty$, there is $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $x_n > \sqrt{|M|}$; Similarly there is $N_2 \in \mathbb{N}$ such that for all $n > N_2$, $y_n > \sqrt{|M|}$. Now take $N = \max\{N_1, N_2\}$. For any $n > N$, we have $x_n y_n > \sqrt{|M|}\, \sqrt{|M|} = |M| \geqslant M$.

- Case 2. We only prove for $a > 0$, $b = \infty$ (The other seven sub-cases can be proved similarly). We need to show that for every $M \in \mathbb{R}$, there is $N \in \mathbb{N}$ such that for all $n > N$, $x_n y_n > M$.

Since $x_n \longrightarrow a > 0$, there is $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $x_n > a/2$. On the other hand, as $y_n \longrightarrow \infty$, there is $N_2 \in \mathbb{N}$ such that for all $n > N_2$, $y_n > 2\,|M|/a$. Now take $N = \max\{N_1, N_2\}$, for every $n > N$ we have $x_n\, y_n > (a/2)\,(2\,|M|/a) = |M| \geqslant M$. □

Example 1.48. The reason why $\infty - \infty$ or $0 \cdot (+\infty)$ or $0 \cdot (-\infty)$ should not be involved is that in those cases the results depend on how ∞ is approached and consequently differ case by case. For example, $1/n^2 \longrightarrow 0$, $n \longrightarrow \infty$, $n^3 \longrightarrow \infty$ but $(1/n^2) \cdot n \longrightarrow 0$ while $(1/n^2) \cdot n^3 \longrightarrow \infty$. So $0 \cdot \infty$ cannot be defined.

As an exercise, examples should be constructed for other cases.

Theorem 1.49. (Extension of Comparison Theorem 1.22) *Let $x_n \longrightarrow a$, $y_n \longrightarrow b$ with a, b extended real numbers. If there is $N_0 \in \mathbb{N}$ such that $x_n \leqslant y_n$ for all $n \geqslant N_0$, then*

$$\lim_{n \longrightarrow \infty} x_n \leqslant \lim_{n \longrightarrow \infty} y_n. \tag{1.63}$$

Proof. Here we only need to deal with three cases: $a = -\infty, a \in \mathbb{R}, a = +\infty$.

- $a = -\infty$. As $-\infty \leqslant b$ for every extended real number b, the proof ends.
- $a \in \mathbb{R}$. If $b \in \mathbb{R}$, the conclusion follows from Theorem 1.22. If $b = \infty$, the conclusion holds. We only need to show that $b = -\infty$ cannot happen. Assume the contrary. As $x_n \longrightarrow a \in \mathbb{R}$, there is $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $x_n > a - 1$. On the other hand, as $y_n \longrightarrow -\infty$, there is $N_2 \in \mathbb{N}$ such that for all $n > N_2$, $y_n < a - 1$. Take $N = \max\{N_1, N_2, N_0\}$. For every $n > N$, we have $y_n < a - 1 < x_n$ while at the same time by assumptions in the theorem, $x_n \leqslant y_n$. Contradiction.
- $a = \infty$. We need to show that $b = -\infty$ or $b \in \mathbb{R}$ would lead to contradiction. The proof is very similar to that of the previous case and is left as exercise. □

Theorem 1.50. (Extension of "Squeeze Theorem" 1.24) *Let $x_n \rightarrow a$ and $y_n \longrightarrow a$ where a is an extended real number. Let $\{w_n\}$ be a sequence. Assume that there is $N_0 \in \mathbb{N}$ such that for all $n \geqslant N_0$,*

$$x_n \leqslant w_n \leqslant y_n. \tag{1.64}$$

then $w_n \longrightarrow a$.

Proof. We only need to prove the following two cases: $a = \infty$ and $a = -\infty$.

- $a = \infty$. For any $M \in \mathbb{R}$, there is $N_1 \in \mathbb{N}$ such that $x_n > M$ for all $n > N_1$. Now take $N = \max\{N_0, N_1\}$. Then we have for all $n > N$, $w_n \geqslant x_n > M$. Therefore $w_n \longrightarrow \infty$.
- $a = -\infty$. For any $M \in \mathbb{R}$ there is $N_1 \in \mathbb{N}$ such that $y_n < M$ for all $n > N_1$. Now take $N = \max\{N_0, N_1\}$. Then we have for all $n > N$, $w_n \leqslant y_n < M$. Therefore $w_n \longrightarrow -\infty$. □

Theorem 1.51. (Extension of Subsequence Lemmas 1.35, 1.36) *Let $\{x_n\}$ be a sequence.*

a) *If $x_n \longrightarrow a$, then every subsequence $\longrightarrow a$;*
b) *If every subsequence converges to the same a, then $x_n \longrightarrow a$.*

Proof. Since $a \in \mathbb{R}$ has already been settled in Lemmas 1.35, 1.36, we only need to deal with $a = \pm\infty$. We only prove for $a = \infty$ here. The case $a = -\infty$ can be proved similarly and is left as exercise.

a) For any given $M \in \mathbb{R}$, there is $N \in \mathbb{N}$ such that for all $n > N$, $x_n > M$. Let $\{x_{n_k}\}$ be an arbitrary subsequence. Since $n_1 < n_2 < n_3 < \cdots$ there is $K \in \mathbb{N}$ such that for all $k > K$, $n_k > N$, and consequently $x_{n_k} > M$ for all $k > K$.
b) We prove by contradiction. Assume that all of its subsequences $\longrightarrow \infty$ but $x_n \not\longrightarrow \infty$. Then there is $M_0 \in \mathbb{R}$ such that for every $N \in \mathbb{N}$ there is $n > N$ such that $x_n \leqslant M_0$. Take $N = 1$, we obtain $x_{n_1} \leqslant M_0$. Now take $N = n_1$, we obtain $n_2 > n_1$ with $x_{n_2} \leqslant M_0$. Doing this again and again we obtain a subsequence x_{n_k} with $x_{n_k} \leqslant M_0$. There are two cases:
    - There is $M \in \mathbb{R}$ such that $x_{n_k} \geqslant M$ for all k. In this case we can apply Bolzano-Weierstrass to obtain a convergent subsequence $x_{n_{k_l}}$. As $M \leqslant x_{n_{k_l}} \leqslant M_0$, the comparison theorem 1.22 leads to $x_{n_{k_l}} \longrightarrow b$ for some $M \leqslant b \leqslant M_0$.

- For every $M \in \mathbb{R}$ there is $k \in \mathbb{N}$ such that $x_{n_k} < M$. Take $M = -1$. We have $x_{n_{k_1}} < -1$. Now take $M = x_{n_{k_1}} - 1$, we have $x_{n_{k_2}} < x_{n_{k_1}} - 1 < -2$. Continue doing this we have a subsequence $x_{n_{k_l}}$ satisfying $x_{n_{k_l}} < -l$. Thus $x_{n_{k_l}} \longrightarrow -\infty$. □

Think: There is no way to extend "Relation to Cauchy sequence" theorem 1.28 here.

Putting all the things we understood so far together, we reach

Theorem 1.52. (Extension of Classification Theorem 1.39) *Let $\{x_n\}$ be a sequence. Then*

- *either $x_n \longrightarrow a$ for some extended real number a, or*
- *there are two subsequences $x^1_{n_k} \longrightarrow b_1, x^2_{n_k} \longrightarrow b_2$ for extended real numbers $b_1 \neq b_2$.*