0. Preparation: Logic, Sets, and Functions

Roughly speaking, at the most basic level, real analysis studies behaviors of real-valued functions defined over real numbers under certain limiting processes.

On one hand, real analysis is straightforward and easy. Many "theorems" are simply rigorous justifications of what we either already know intuitively (for example: a function takes intermediate values) or can convince ourselves in a few minutes (for example: the mean value theorem); On the other hand, however, real analysis is subtle and difficult. All these "simple intuitive facts" have hidden pitfalls and caveats which makes their mathematical proofs surprisingly difficult. This bipolarity makes analysis one of the hardest mathematical courses, there is always the danger of getting carried away by intuition and reaching wrong conclusions.

Therefore we need to walk the fine line between rigor and intuition. To be able to do that, we need to start from the rigor side, getting familiar with logic and sets.

0.1. Logic.

Everyday language thrives on ambiguity, while mathematics tolerates little of it. Therefore it is necessary for mathematics to introduce its own "language" to facilitate formal reasoning.

- True and False.

  In basic mathematical logic, every statement is either true or false. There is no middle ground. For example

  If $x > 9$, then $x^2 - 3x > 10$.

  is a true statement. To show a statement is true, we need a **proof**: A rigorous chain of logical reasoning starting from a known true fact and end at the claim to be proved. For this claim, we can prove as follows:

  Proof. Since $x > 9$, we have $x - 5 > 0$, $x + 2 > 0$. Therefore

  $$(x-5)(x+2) > 0. \tag{0.1}$$

  As $(x-5)(x+2) = x^2 - 3x - 10$, we have $x^2 - 3x - 10 > 0$ which gives $x^2 - 3x > 10$. ☐

  Of course more often proofs are written in a way that is more consistent with human thinking. For example, the above proof would most likely be written like:[0.1]

  Proof. First notice that to prove $x^2 - 3x > 10$, we only need to prove $x^2 - 3x - 10 > 0$. This is equivalent as $(x-5)(x+2) > 0$. But as $x > 9$, we have both $x - 5, x + 2 > 0$. Thus the proof ends. ☐

  On the other hand,

  The square of a real number is always bigger than 0

  is a false statement. To show a statement is false, we need a **counter-example**. In this case, we need one single real number whose square is not bigger than 0.

  Remark 0.1. Constructing counter-examples may sound easy, but it often is not. Furthermore, in some sense, the only way to gain understanding of real analysis is through the following process:

  > When you see a theorem, look at its assumptions. Take one away. Construct a counter-example to show that this theorem is false without that assumption. Put that one back and take another away. Do the same. Stop when you have constructed at least one counter-example for every assumption. If you cannot do this for any of the assumptions, you don't understand the theorem at all.

- "and", "or", "not".

0.1. A kind of cynical statement about "proofs" is the following: To obtain a proof, you start from both the known fact and the claim needs to be proved, and try to bring them together (in the above proof we "brought them together" at $(x-5)(x+2) > 0$). If you cannot bring them together, do as much as you can and put "obviously" in between.

In mathematics, "A and B" means roughly the same thing as in everyday language. It is true when both A and B are true, and is false in all other cases. For example,

$$5>3 \text{ and } 6>4 \tag{0.2}$$

is a true statement, while

$$3>5 \text{ and } 5>2 \tag{0.3}$$

is false.

On the other hand, "A or B" in everyday language has two meanings: "one or the other, but not both", and "one or the other or both". In mathematics the latter meaning is adopted. Thus

$$3>5 \text{ or } 5>2 \tag{0.4}$$

is a true statement while

$$3>5 \text{ or } 1>2 \tag{0.5}$$

is false.

"Not A", denoted $\sim A$, means the opposite of A, that is when A is true, $\sim A$ is false and vice versa.

More precisely, we assign a "truth value" of T (for true) and F(for false) to every statement. The truth values of concatenation of statements by "and" and "or" are defined as follows:

A	B	A and B	A or B	˜A (Not A)
T	T	T	T	F
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Table 0.1. Truth table of "and", "or", "not".

Such formal definition may seem an overkill for simple expressions like $5>3 \,\text{and}\, 6>4$, but it helps us to unravel more complicated expressions.

Example 0.2. Is the claim

$$[(5\,\text{is prime})\,\text{and}\,(5>7)]\,\text{or}\,[\sim(4>5)] \tag{0.6}$$

true or false?[0.2]

Solution. After judging the truth value of each inner-most claim, we reach

$$\begin{aligned}[(5\,\text{is prime})\,\text{and}\,(5>7)]\,\text{or}\,[\sim(4>5)] &= (T \text{ and } F) \text{ or } (\sim F)\\ &= F \text{ or } T\\ &= T. \end{aligned} \tag{0.7}$$

So the statement is true.

Such formality also helps discovering equivalent statements.

Example 0.3. Show that $\sim(A \text{ or } B)$ is equivalent to $(\sim A)$ and $(\sim B)$.

Proof. All we need to do is to show that the two statements have the same truth table. To construct such a table we need to replace A, B by all four cases $T, T; T, F; F, T; F, F$ and calculate the truth values. For example, in the case T, T, we have

$$\sim(A \text{ or } B) = \sim(T \text{ or } T) = \sim T = F; \tag{0.8}$$

$$(\sim A) \text{ and } (\sim B) = (\sim T) \text{ and } (\sim T) = F \text{ and } F = F. \tag{0.9}$$

After some calculation we conclude

A	B	$\sim(A\,\text{and}\,B)$	$\sim A$ and $\sim B$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

0.2. Yes I know this one is not complicated at all... I need more time to think of a truly complicated one. Please tell me if you have one. I will be grateful.

We see that both statements have exactly the same truth values and the proof ends. □

- Implications.

  Proofs in mathematics usually consist of a long chain of “if A is true, then B is true” or “B is true only if A is true” or “since A is true, B is true” or “because A is true, B is true” or “A implies B”... All these – first convince yourself that they all mean the same! – can be simply written as

$$A \Longrightarrow B. \tag{0.10}$$

Formally, the definition of $A \Longrightarrow B$ is given by

$$A \Longrightarrow B := \sim A \text{ or } B. \tag{0.11}$$

Remark 0.4. Some calculation soon reveals the following deviation from common sense: If A is false, then $A \Longrightarrow B$ is true no matter what B is. For example, the statement “if 5 is even then $x^2 + 1 < 0$” is a true statement. Such implication is called “vacuously true”. It turns out that when dealing with meaningful problems, this caveat does not matter.

Remark 0.5. The most important thing to understand about implication is the equivalence of $A \Longrightarrow B$ and $\sim B \Longrightarrow \sim A$. In other words, if we want to show “if A is true, then B is true”, it is enough to show “if B is not true, then A is not true”. This is the logical foundation of proof by contradiction (reductio ad absurdum).

Example 0.6. Prove by contradiction that $\sqrt{2}$ is irrational, that is cannot be written as p/q with integers p, q.

Proof. Here clearly $B = {}''\sqrt{2}$ is irrational$''$, while A is not explicitly given. Rigorously speaking, $A =$ All true statements about rational numbers. Thus to prove by contradiction is to show that

$$\text{If } {}''\sqrt{2} \text{ is irrational}'' \text{ is not true, then some true statement about rational numbers is false.} \tag{0.12}$$

or equivalently,

$$\text{If } \sqrt{2} \text{ is rational, then some true statement about rational numbers has to be false.} \tag{0.13}$$

Therefore, to start our proof, we assume $B = \sqrt{2} = p/q$ with p, q integers. It is clear that we can assume p, q to be natural numbers, and at least one of them is odd. Now taking square of the equation we have

$$2 = p^2/q^2 \Longrightarrow p^2 = 2\,q^2. \tag{0.14}$$

If p is odd, we have already reached a contradiction; On the other hand, if p is even, by our assumption q must be odd. Write $p = 2\,k$ we have

$$(2\,k)^2 = 2\,q^2 \Longrightarrow q^2 = 2\,k^2 \tag{0.15}$$

and reach contradiction again. □

- Equivalence.

  If $A \Longrightarrow B$ and $B \Longrightarrow A$, then we say A and B are logically equivalent, denoted $A \Longleftrightarrow B$. Logical equivalence means A, B are always both true or both false. Establishing such equivalence (preferably between one easy/intuitive fact and one hard/non-intuitive fact) is one of the most important tasks of real analysis.

Example 0.7. Calculate the truth table of $A \Longleftrightarrow B$.
Solution. We consider the case $A = F$, $B = F$ and omit the detailed calculation of the other three cases. In this case we have

$$\begin{aligned} A \Longleftrightarrow B = A \Longrightarrow B \text{ and } B \Longrightarrow A &= (F \Longrightarrow F) \text{ and } (F \Longrightarrow F) \\ &= T \text{ and } T \\ &= T. \end{aligned} \tag{0.16}$$

Finally we have the following table

A	B	$A \Longleftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

This time it's quite consistent with common sense!

Remark 0.8. Since $A \Longrightarrow B$ means "A only if B" and $B \Longrightarrow A$ means "A if B", $A \Longleftrightarrow B$ is often written as "A if and only if B" or "A iff B".

Remark 0.9. When proving $A \Longleftrightarrow B$, usually one has to prove $A \Longrightarrow B$ and $B \Longrightarrow A$ separately.

- Quantifiers.

  We often use $\forall$ to denote "for every", and $\exists$ to denote "there exists".[0.3] Thus

$$\forall x\ \exists y, \qquad y > x. \tag{0.17}$$

means

$$\text{For every } x, \text{ there exists } y, \text{ such that } y > x. \tag{0.18}$$

Example 0.10. The statement $\forall x \exists y,\ y > x$ is true, as we can take $y = x + 1$. On the other hand $\exists y \forall x,\ y > x$ is false. Because if there is such a y, we can simply take $x = y + 1$ and reveal the absurdity of the claim.

- Proof by contradiction with quantifiers.

  Quantifiers abound in real analysis. For example, a function $f(x)$ is continuous at a point x_0 if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every x with $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$. In symbols:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \text{ with } |x - x_0| < \delta, \qquad |f(x) - f(x_0)| < \varepsilon. \tag{0.19}$$

When quantifiers are involved, the initial assumption in proof by contradiction gets quite tricky to set up. The rule here is that we should switch $\forall$ and $\exists$, and put "Not" to the final conclusion. For example, if we want to show $\forall x,\ B(x)$ by contradiction, our starting assumption should be $\exists x,\ \sim B(x)$.

As an example, if we would like to prove by contradiction that $f(x)$ is continuous at x_0, we should assume[0.4]

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x \text{ with } |x - x_0| < \delta, \qquad |f(x) - f(x_0)| \geqslant \varepsilon \tag{0.20}$$

and expect to reach some absurd conclusion later.

0.3. Clearly $\exists$ comes from the first letter "E". On the other hand $\forall$ comes from "Any", but "for any" is ambiguous.

0.4. The "x with $|x - x_0| < \delta$" may cause some confusion here. A better way of writing this is $x \in (x_0 - \delta, x_0 + \delta)$ which will be introduced later.

0.2. Sets.

0.2.1. Definition.

Definition 0.11. (Set) *A set is a collection of objects. These objects are called "elements" of the particular set.*

A set is often denoted as follows:

$$\{...\} \tag{0.21}$$

Here $\{\}$ means it's a set, and "$...$" is a characterization of its elements.

If an object a is an element of a set A, we write

$$a \in A, \tag{0.22}$$

if a is not an element of A, we write

$$a \notin A. \tag{0.23}$$

Example 0.12. The following are examples of sets.

- Let $A = \{$All the humans that ever walked the earth$\}$. We have

$$\text{Barack Obama} \in A, \qquad \text{Jonathan Osterman (aka Dr. Manhattan)} \notin A. \tag{0.24}$$

- Let $B = \{$All the real numbers that satisfy $|x-1| < 3\}$. Then

$$1 \in B, \qquad 4 \notin B. \tag{0.25}$$

- Let $C = \{$All natural numbers n such that there are integers solving $x^n + y^n = z^n\}$. Thanks to Andrew Wiles we now know that

$$1, 2 \in C, \qquad 3, 4, ... \notin C. \tag{0.26}$$

- Let $D = \{x \in B: x^2 = 9\}$. Then $3 \in D$, $-3 \notin D$.

Example 0.13. There are several special sets of numbers that are so important that they have special letters assigned to them.

- The set of natural numbers is denoted $\mathbb{N}$; So $\mathbb{N} = \{0, 1, 2, ...\}$.
- The set of integers is denoted $\mathbb{Z}$; So $\mathbb{Z} = \{..., -1, 0, 1, 2, ...\}$.
- The set of rational numbers is denoted $\mathbb{Q}$;
- The set of real numbers is denoted $\mathbb{R}$;
- The set of complex numbers is denoted $\mathbb{C}$.

Example 0.14. (Empty set) There is a special set called "empty set", denoted $\varnothing$, which is defined as a set with no element. In other words, there is no object a satisfying $a \in \varnothing$, or equivalently, every object a satisfies $a \notin \varnothing$.

Note. (Banach-Tarski Paradox) In case you get bored now, here is a fun fact about the Platonic universe of mathematics. Take a solid ball, treat it as a set of points in $\mathbb{R}^3$, the 3-dimensional space. You can break it apart into several pieces (subsets of points in space), and then re-assemble these pieces into two solid balls that are identical to the original one. Of course the number two is not important here – you can make three, four, five, or even more by repeating the process.

This is one way (totally overkill!) to show the existence of sets that cannot be assigned a volume.

Maybe the most surprising thing about this paradox is that such break-apart/reassmble would not be possible if we do not assume the following (totally innocent looking) Axiom of Choice:

Given a bunch of sets, there is a new set consisting of exactly one element from each set.

0.2.2. Relations between sets.

Two sets A, B can have the following possible relations:

- Subset. If every element of A is also an element of B, we say A is a subset of B, denoted $A \subseteq B$. Using the logical statements in the last subsection, $A \subseteq B$ is defined as

$$x \in A \Longrightarrow x \in B. \tag{0.27}$$

  For example $\mathbb{N} \subseteq \mathbb{Z}$.

- Equal. If A, B have exactly the same elements, we say $A = B$. For example, if $A := \{x \in \mathbb{R}: x^2 + 1 = 0\}$, then $A = \varnothing$. On the other hand, if $B := \{x \in \mathbb{C}: x^2 + 1 = 0\}$, then $B = \{i, -i\}$.

  $A = B$ can be defined as

$$x \in A \Longleftrightarrow x \in B \tag{0.28}$$

  which immediately gives

$$A = B \text{ is equivalent to } [(A \subseteq B) \text{ and } (B \subseteq A)]. \tag{0.29}$$

  When the relation between A, B are not obvious, showing $A \subseteq B$ and then $B \subseteq A$ is most likely the only way to prove $A = B$.

- Proper subset. From the above definitions we know that if $A = B$, then $A \subseteq B$. However, when we talk about subsets, often we do not mean this trivial situation. Thus we define "A is a proper subset of B", denoted $A \subset B$, if $A \subseteq B$ but $A \neq B$.

Proposition 0.15. *We have*

- *$\varnothing \subseteq A$ for any set A.*
- *Let A be a set. If $A \subseteq \varnothing$, then $A = \varnothing$.*
- *$A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*

Proof. *The first statement requires us showing*

$$x \in \varnothing \Longrightarrow x \in A. \tag{0.30}$$

As $x \in \varnothing$ is always false, the statement is always true.

For the second statement, as $A \subseteq \varnothing$ is defined as

$$x \in A \Longrightarrow x \in \varnothing \tag{0.31}$$

and $x \in \varnothing$ is always false, $x \in A$ must be false for all x too. Consequently $A = \varnothing$.

For the last statement, to show that

$$x \in A \Longrightarrow x \in C \tag{0.32}$$

is true, we only need to show that when $x \in A$ is true, $x \in C$ must be true, since when $x \in A$ is false the statement is automatically true.[0.5]

Since

$$[x \in A \Longrightarrow x \in B \text{ and } x \in B \Longrightarrow x \in C] \tag{0.33}$$

is true, we must have both $x \in A \Longrightarrow x \in B$, $x \in B \Longrightarrow x \in C$ true. Since $x \in A$ is true, the truth of $x \in A \Longrightarrow x \in B$ implies $x \in B$ is true, which together with the truth of $x \in B \Longrightarrow x \in C$ gives that $x \in C$ is true. ☐

Remark 0.16. One can also prove through words instead of formal logic. For example, the last statement can be proved as follows:

0.5. It is actually quite common sense. To see whether A is part (subset) of C, we don't need to worry about those x that does not belong to A, whether they are in C or not is irrelevant.

Proof. We consider two cases.

If $A = \varnothing$, then by the first claim (already proved!) $A \subseteq C$; Otherwise, all we need to show is that every $x \in A$ also $\in C$. Take any $x \in A$. Since $A \subseteq B$, we have $x \in B$. Now $B \subseteq C$ gives $x \in C$. This ends the proof.

One thing to remember is that you almost always need to discuss separately the cases where one or another set is the empty set. Since if $A = \varnothing$, then "take any $x \in A$" is meaningless.

0.2.3. Set operations.

Given two sets A, B, the following operations can be carried out.

- Union. The union of two sets A, B is the new set obtained from putting all elements of A and all elements of B together. We denote

$$A \cup B := \{x : x \in A \text{ or } x \in B\}. \tag{0.34}$$

  Recall that here we are using the mathematical "or" which includes the case $x \in$ both A, B.

- Intersection. The intersection of A, B is the set of all common elements of A, B:

$$A \cap B := \{x : x \in A \text{ and } x \in B\}. \tag{0.35}$$

- Set difference. The "set difference of B from A" is the set of all elements that are in A but not in B:

$$A \backslash B := \{x : x \in A, x \notin B\}. \tag{0.36}$$

  It is also called the "complement of B relative to A". In some textbooks the notation $A - B$ is also used.

- Complement. When the set A is an "ambient" set for the discussion, $A \backslash B$ is often denoted as B^c and called "complement of B". For example, if all our discussion is about real numbers, then $\mathbb{R} \backslash B$ is often denoted B^c, with the ambient set $\mathbb{R}$ implicit.

Proposition 0.17. *The following are very useful in proving set relations.*

a) $A \subseteq C, B \subseteq C \Longrightarrow A \cup B \subseteq C$;

b) $C \subseteq A, C \subseteq B \Longrightarrow C \subseteq A \cap B$;

c) $A \subseteq B \Longrightarrow A \cap C \subseteq B \cap C$;

d) $A \subseteq B \Longrightarrow A \cup C \subseteq B \cup C$.

e) $A \subseteq B \Longrightarrow C \backslash B \subseteq C \backslash A$.

Proof.

a) We are given

$$x \in A \Longrightarrow x \in C \text{ and } x \in B \Longrightarrow x \in C \tag{0.37}$$

and need to show

$$(x \in A \text{ or } x \in B) \Longrightarrow x \in C. \tag{0.38}$$

We only need to deal with the case $x \in A$ or $x \in B$ is true. Thus at least one of $x \in A$,$x \in B$ is true. Say $x \in A$ is true. Then $x \in A \Longrightarrow x \in C$ gives $x \in C$ is true. Similarly if $x \in B$ is true we also conclude $x \in C$ is true. Consequently if $x \in A$ or $x \in B$ is true then $x \in C$ is true, which is exactly

$$(x \in A \text{ or } x \in B) \Longrightarrow x \in C. \tag{0.39}$$

b)–e) can be proved similarly and is omitted. $\square$

Theorem 0.18. (Properties of set operations) *Let A, B, C be sets*

a) $A \cap B \subseteq A \subseteq A \cup B$.

b) $A\cup A=A$; $A\cap A=A$.

c) $A\cup B=B\cup A$; $A\cap B=B\cap A$.

d) $(A\cup B)\cup C=A\cup(B\cup C)$; $(A\cap B)\cap C=A\cap(B\cap C)$.

e) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$; $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$.

Proof. The proofs are quite similar, thus we will not give all the details but only prove the first half of e).

Recall that to prove $A=B$, all we need are $A\subseteq B$ and $B\subseteq A$.

- $[(A\cap B)\cup(A\cap C)]\subseteq A\cap(B\cup C)$. Since $B\subseteq B\cup C$, application of Proposition 0.17 gives $A\cap B\subseteq A\cap(B\cup C)$. Similarly we have $A\cap C\subseteq A\cap(B\cup C)$. Applying Proposition 0.17 again, we have
$$[(A\cap B)\cup(A\cap C)]\subseteq A\cap(B\cup C). \tag{0.40}$$
- $A\cap(B\cup C)\subseteq[(A\cap B)\cup(A\cap C)]$. This is a bit tricky. First consider the case $A\cap(B\cup C)=\varnothing$. In this case $A\cap(B\cup C)\subseteq$ any other set, so the claim holds.

  Otherwise, for every $x\in A\cap(B\cup C)$, by definition $x\in A$ and $x\in B$ or C. We discuss all three cases. If $x\in B$, then $x\in A\cap B\subseteq(A\cap B)\cup(A\cap C)$; If $x\in C$, then $x\in A\cap C\subseteq(A\cap B)\cup(A\cap C)$. ☐

Remark 0.19. A good tool to understand relations of sets is the Venn graph, or Venn diagram (the wiki page is good enough). However, drawing a Venn graph is **NOT A PROOF** of set relations. It is just a help of visualizing what is going on.

0.2.4. A glimpse of infinity.

The operations $\cup,\cap$ can be generalized naturally to involve more than two or even infinitely many sets. For example we can define
$$A\cap B\cap C=\{x: x\in A \text{ and } x\in B \text{ and } x\in C\}. \tag{0.41}$$
More specifically, let $\mathcal{E}=\{E_\alpha\}_{\alpha\in A}$ be a collection of sets. Then

i. The union of $\mathcal{E}$ is
$$\cup_{\alpha\in A}E_\alpha:=\{x: x\in E_\alpha \text{ for at least one } \alpha\in A\}; \tag{0.42}$$
ii. The intersection of $\mathcal{E}$ is
$$\cap_{\alpha\in A}E_\alpha:=\{x: x\in E_\alpha \text{ for all } \alpha\in A\}. \tag{0.43}$$

Example 0.20. Let $E_a=\{x\in\mathbb{R}: x<1/a\}$ for $a\in A=\{a\in\mathbb{R}: a>1\}$. Calculate (meaning: give the simplest description possible, preferably one single formula) $\cap_{a\in A}E_a$ and $\cup_{a\in A}E_a$.

Solution.

- $\cap_{a\in A}E_a$. Solving such problems usually involve three steps.
  1. Determine the answer. If $a>1$, then $0<1/a$. So $\{x: x\leqslant 0\}\subseteq E_a$ for all a. As a gets larger, $1/a$ gets smaller so we are pretty sure $E:=\{x: x\leqslant 0\}$ should be the answer. What we need to do now is to show $\cap_{a\in A}E_a=E$. Remember our only method of showing equality of two sets?
  2. $\cap_{a\in A}E_a\subseteq E$. We need to show that if $x\in\mathbb{R}$ satisfies $x<1/a$ for all $a>1$, then $x\leqslant 0$. We prove by contradiction. Assume there is $x>0$ such that $x<1/a$ for all $a>1$. Set $a=\frac{1+x}{x}>1$. Then $1/a=\frac{x}{1+x}<x$ as $x>0$, contradiction.[0.6]
  3. $E\subseteq\cap_{a\in A}E_a$. We need to show that every $x\leqslant 0$ satisfies $x<1/a$ for every $a>1$. This is obvious as $x\leqslant 0<1/a$.
- $\cup_{a\in A}E_a$. The procedure is similar, we get $\cup_{a\in A}E_a=\{x: x<1\}$.

0.6. Think: Why not just set $a=1/x$? Then $1/a=x\geqslant x$, contradiction!

0.3. Functions.

0.3.1. Definitions.

Definition 0.21. (Functions) *A function $f: A \mapsto B$ is a special relation between two sets A and B: it relates to every $a \in A$ exactly one element in B, this element is denoted $f(a)$. We call A the domain of the function and B the range of the function.*

Example 0.22. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d, e\}$. Let a relation be given by

$$\mathcal{R} = \{(1, a), (2, c), (3, b), (4, d), (5, d)\}. \tag{0.44}$$

then this is a function with domain A and range B.

On the other hand, the relation

$$\mathcal{R}' = \{(1, a), (1, b), (2, c), (3, e)\} \tag{0.45}$$

fails to be a function because

- It does not relate an element in B to **every** $a \in A$;
- For some $a \in A$ it relates more than one element in B.

Remark 0.23. It is important to keep in mind that a function is a triplet: Domain A, Range B, Relation $\mathcal{R}$ (or f). Changing any one of the three leads to a different function. Rigorously speaking, $\sin x$ over $(-1, 1)$ and $\sin x$ over $(-2, 2)$ are two different function, they just happen to coincide over $(-1, 1)$. In real analysis this may seem like some annoying triviality, but this understanding is extremely important when studying Complex Analysis, Functional Analysis, and many other higher level analysis courses.

From the above we see that every $a \in A$ is involved, while some elements in B may not be involved. To make things clear we define

Definition 0.24. (Image) *Let A, B be sets and let $f: A \mapsto B$ be a function. The image of a subset $S \subseteq A$ under f is defined as*

$$f(S) := \{b \in B : \exists a \in S \text{ such that } f(a) = b\}. \tag{0.46}$$

The image of f is defined as the special case

$$\text{Image}(f) := f(A) = \{b \in B : \exists a \in A \text{ such that } f(a) = b\}. \tag{0.47}$$

Example 0.25. Let $A = B = \mathbb{R}$, $S = \{a \in \mathbb{R} : 0 < a < \pi\}$, $f = \sin$. Then we have

$$\text{Image}(f) = \{x \in \mathbb{R} : -1 \leqslant x \leqslant 1\}. \tag{0.48}$$

$$f(S) = \{x \in \mathbb{R} : 0 < x \leqslant 1\}. \tag{0.49}$$

From this example we see that the image of f may only be a proper subset of B.

When studying functions, it is often important to study those $a \in A$ such that $f(a)$ has certain property. More precisely, we need a notation for those $a \in A$ such that $f(a)$ belongs to a certain subset $S \subseteq B$.

Definition 0.26. (Pre-image) *Let A, B be sets and let $f: A \mapsto B$ be a function. The pre-image of a subet $S \subseteq B$ is defined as*

$$f^{-1}(S) := \{a \in A : \exists b \in S \text{ such that } f(a) = b\}. \tag{0.50}$$

Example 0.27. Let $A = B = \mathbb{R}$, $f = \sin$, $S = \{1\}$. Then

$$f^{-1}(S) = 2 k \pi + \frac{\pi}{2}, \qquad k \in \mathbb{Z}. \tag{0.51}$$

If we let $S = \{2\}$, then clearly

$$f^{-1}(S) = \varnothing. \tag{0.52}$$

We see that the pre-image of a single element may not be a single element, it may also be empty or contain more than one element.

0.3.2. When functions meet sets.

Lemma 0.28. *Let $f: X \mapsto Y$ be a function. Let $A, B \subseteq X$ and $S, T \subseteq Y$. Then the following holds.*

a) *If $A \subseteq B$ then $f(A) \subseteq f(B)$.*

b) *If $S \subseteq T$ then $f^{-1}(S) \subseteq f^{-1}(T)$.*

Proof. Left as exercise. □

Remark 0.29. Note that $A \subset B$ ($S \subset T$) does not imply $f(A) \subset f(B)$ ($f^{-1}(S) \subset f^{-1}(T)$). The construction of counterexamples are left as exercises.

Theorem 0.30. *Let $f: X \mapsto Y$ be a function. Let $A, B \subseteq X$ and $S, T \subseteq Y$. Then*

a) $f(A \cap B) \subseteq f(A) \cap f(B)$.

b) $f(A \cup B) = f(A) \cup f(B)$.

c) $f(A \backslash B) \supseteq f(A) \backslash f(B)$.

d) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

e) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.

f) $f^{-1}(S \backslash T) = f^{-1}(S) \backslash f^{-1}(T)$.

Proof.

a) Since $A \cap B \subseteq A$, Lemma 0.28 gives $f(A \cap B) \subseteq f(A)$; Application of the same lemma to $A \cap B \subseteq B$ gives $f(A \cap B) \subseteq f(B)$. Therefore $f(A \cap B) \subseteq f(A) \cap f(B)$.

b) We need to show $f(A \cup B) \subseteq f(A) \cup f(B)$ and $f(A) \cup f(B) \subseteq f(A \cup B)$.

- $f(A \cup B) \subseteq f(A) \cup f(B)$. Take any $y \in f(A \cup B)$. Then there is $x \in A \cup B$ such that $y = f(x)$. Now $x \in A \cup B$ has two cases: $x \in A$ and $x \in B$. In the first case we have $y \in f(A)$ and in the second we have $y \in f(B)$. Therefore $x \in A \cup B$ implies $y \in f(A) \cup f(B)$. So $f(A \cup B) \subseteq f(A) \cup f(B)$.
- $f(A) \cup f(B) \subseteq f(A \cup B)$. Since $A \subseteq A \cup B$, application of Lemma 0.28 gives $f(A) \subseteq f(A \cup B)$. Application of the same lemma to $B \subseteq A \cup B$ gives $f(B) \subseteq f(A \cup B)$. Therefore $f(A) \cup f(B) \subseteq f(A \cup B)$.

c) Left as exercise.

d) We need to show $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$ and $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$.

- $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$. As $S \cap T \subseteq S$, application of Lemma 0.28 gives $f^{-1}(S \cap T) \subseteq f^{-1}(S)$. The same lemma applied to $S \cap T \subseteq T$ gives$f^{-1}(S \cap T) \subseteq f^{-1}(T)$. Therefore $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$.
- $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$. Take any $x \in f^{-1}(S) \cap f^{-1}(T)$. As $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S)$, we have $x \in f^{-1}(S)$ so $f(x) \in S$. On the other hand, $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(T)$ gives $f(x) \in T$. Therefore $f(x) \in S \cap T$ which means $x \in f^{-1}(S \cap T)$. Thus ends the proof for $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$.

e) Left as exercise.

f) Left as exercise. □

0.3.3. Inverse functions.

In analysis it is often useful to switch the roles of A, B and think of f as a function from B to A. There is we would like to find a function $g: B \mapsto A$ such that

$$g(f(a)) = a \text{ for every } a \in A; \qquad f(g(b)) = b \text{ for every } b \in B. \tag{0.53}$$

Such a function is called an inverse function of f. However as we have seen, for general function this is not possible. Now let's see what extra conditions we need.

Definition 0.31. (one-to-one,onto,bijection) *Let A, B be sets and $f: A \mapsto B$ a function.*

We say f is one-to-one if whenever $f(a_1) = f(a_2)$, we have $a_1 = a_2$.

We say that f is onto if whenever $b \in B$ there exists $a \in A$ such that $f(a) = b$.

We say f is a bijection if it is both one-to-one and onto.

Example 0.32. Consider the following functions: $A = B = \mathbb{R}$,

$$f_1(x) = 2\,x + 4; \quad f_2(x) = \arctan x; \quad f_3(x) = \sin x; \quad f_4(x) = 2\,x^3 + x^2 + 12\,x + 4. \tag{0.54}$$

Then f_1 is one-to-one and onto, f_2 is one-to-one but not onto, f_3 is neither, f_4 is onto but not one-to-one. We give proof to the claim about f_2 and leave others as exercise.
Proof of f_2 being one-to-one but not onto.

To prove that f_2 is one-to-one, we need to show that whenever $f_2(a_1) = f_2(a_2)$, we must have $a_1 = a_2$. One way to show this is through Fundamental Theorem of Calculus:

$$f_2(a_2) - f_2(a_1) = \int_{a_1}^{a_2} f_2'(x)\,\mathrm{d}x = \int_{a_1}^{a_2} \frac{1}{1 + x^2}\,\mathrm{d}x. \tag{0.55}$$

Now if $a_2 > a_1$, from the above we have $f_2(a_2) > f_2(a_1)$; If $a_2 < a_1$, we have $f_2(a_2) < f_2\,(a_1)$. Therefore if $f_2(a_1) = f_2(a_2)$, we must have $a_1 = a_2$.

To show that f_2 is not onto, all we need is a counter-example. That is all we need is one $b \in B = \mathbb{R}$ such that there is no $a \in A = \mathbb{R}$ such that $f_2(a) = b$. This is easy. For example $b = 3$.

Note 0.33. Is the following "proof" of f_2 being one-to-one correct?

If $\arctan a_1 = \arctan a_2$, taking $\tan$ gives $a_1 = a_2$.

Theorem 0.34. *f has an inverse function if and only if f is a bijection.*

Proof. We need to prove

$$f \text{ has an inverse function} \Longleftrightarrow f \text{ is a bijection.} \tag{0.56}$$

Recall that we need to prove $\Longrightarrow$ and $\Longleftarrow$.

- $\Longrightarrow$. Let g be an inverse function of f. As for every $b \in B$, $f(g(b)) = b$, f is onto. Now we show f is one-to-one. Let $f(a_1) = f(a_2)$. As g is a function, it maps $f(a_1) = f(a_2)$ to a single element in a:

$$a_1 = g(f(a_1)) = g(f(a_2)) = a_2. \tag{0.57}$$

- $\Longleftarrow$. Since f is a bijection, for every $b \in B$ there is a unique $a \in A$ such that $f(a) = b$. We define

$$g(b) = a. \tag{0.58}$$

Thus automatically $g(f(a)) = a$. On the other hand, $f(g(b)) = f(a) = b$. So g is an inverse function. □