MATH 217 – Midterm test

27 October 2011

Name: **Student ID:**

- ▷ Time allowed: 50 minutes.
- ▷ Total possible marks: 30. Your *four* best problems determine your marks.
  Bonus marks may be given for your work on the other problems.
- ▷ This is a closed book test!
- ▷ NO calculators, mobile phones, iPods etc.!
- ▷ Good luck!

Problem 1

[7.5 marks]

Given any $t \in \mathbb{R}$, define a sequence (a_n) in $\mathbb{R}$ recursively as

$$a_1 := t \qquad \text{and} \qquad a_{n+1} := a_n(2 - a_n) \quad (n \in \mathbb{N}) .$$

(i) Show that (a_n) is convergent whenever $t \in [0, 2]$, and find $\lim_{n\to\infty} a_n$ in this case. (The limit may depend on t.)

(ii) What can you say about (a_n) if $t \notin [0, 2]$?

Problem 2

[7.5 marks]

Consider the following two statements about any function $f : \mathbb{R} \to \mathbb{R}$:

(i) f is continuous;

(ii) $\text{graph} f := \{x \in \mathbb{R}^2 : x_2 = f(x_1)\}$ is a closed subset of $\mathbb{R}^2$.

Turn (i)☺(ii) into a true logical statement by replacing ☺ with either $\Leftarrow$, $\Rightarrow$, or $\Leftrightarrow$. If your choice is $\Leftarrow$ (resp. $\Rightarrow$) rather than $\Leftrightarrow$, give an example for which $\Rightarrow$ (resp. $\Leftarrow$) is false.

Problem 3

[7.5 marks]

Let A, B be two subsets of $\mathbb{R}^d$, and recall that $A + B = \{a + b : a \in A, b \in B\}$. (If A or B are empty then $A + B := \varnothing$.) Prove or disprove each of the following three statements:

(i) If A and B are closed then $A + B$ is closed;

(ii) If A and B are compact then $A + B$ is compact;

(iii) If A and B are convex then $A + B$ is convex.

Problem 4

[7.5 marks]

Prove *Apollonius' identity*:

$$|c-a|^2 + |b-a|^2 = \frac{1}{2}|c-b|^2 + 2\left|\frac{b+c}{2} - a\right|^2 \quad \forall a, b, c \in \mathbb{R}^d\,.$$

Problem 5

[7.5 marks]

Let $a, b, c \in \mathbb{R}^d$. Demonstrate that the following two statements are equivalent:

(i) $|c-a| = |c-b| + |b-a|$;

(ii) b lies on the line segment from a to c, i.e. $b = (1-t)a + tc$ for some $t \in [0,1]$.

Problem 6

[7.5 marks]

Let $A \subset \mathbb{R}^d$, and assume that $f : A \to \mathbb{R}^m$ is continuous. Prove or disprove each of the following two statements:

(i) If A is closed then $f(A)$ is closed;

(ii) If A is bounded then $f(A)$ is bounded.