MATH 217 – Final exam

19 December 2011

Name: **Student ID:**

- ▷ Time allowed: 3 hours.
- ▷ Total possible marks: 40. All problems have equal weight. Your *six* best problems determine your marks. Bonus marks *may* be given for your work on the other problems.
- ▷ This is a closed book exam!
- ▷ NO calculators, mobile phones, iPods etc.!
- ▷ Good luck!

Problem 1

Consider the set $A := \{x \in \mathbb{R}^3 : \sqrt{|x_1|} + \sqrt{|x_2|} + \sqrt{|x_3|} \leq 1\}$.

(i) Show that A is compact and star-shaped, but not convex.

(ii) Explain why A is a Jordan set, i.e. $A \in \mathcal{J}^3$, and determine $\mu(A)$.

Problem 2

For $d \in \mathbb{N} \setminus \{1\}$, consider the following two statements about any set $B \subset \mathbb{R}^d$:

(i) B is path-connected;

(ii) For every continuous function $f : \mathbb{R}^d \to \mathbb{R}$, the set $f(B)$ is path-connected.

Turn (i)♣(ii) into a true logical statement by replacing ♣ with either $\Leftarrow$, $\Rightarrow$, or $\Leftrightarrow$. If your choice is $\Leftarrow$ (resp. $\Rightarrow$) rather than $\Leftrightarrow$, give an example for which $\Rightarrow$ (resp. $\Leftarrow$) is false.

Problem 3

Let $f : [0, 1] \to \mathbb{R}$ be continuous, and define $g : (0, 1) \to \mathbb{R}$ as

$$g(r) := \int_r^1 \frac{f(x)}{x}\,\mathrm{d}x\,.$$

(i) Demonstrate that $\lim_{r\to 0} g(r)$ may not exist.

(ii) Show that $\lim_{r\to 0} \dfrac{g(r)}{\ln r}$ exists and determine its value.

Problem 4

For the set $C = \{x \in \mathbb{R}^3 : x_3 = 1 - x_1^2 - x_2^2 \geq 0\}$ determine $\alpha := \min_{x \in C} |x|$ and $\beta := \max_{x \in C} |x|$, and identify the sets $\{x \in C : |x| = \alpha\}$ and $\{x \in C : |x| = \beta\}$.

Problem 5

Prove or disprove: If $D \subset \mathbb{R}^d$ is compact and $f : D \to \mathbb{R}^m$ is continuous and one-to-one, then

$$h : \left\{ \begin{array}{ccc} f(D) & \to & D \\ f(x) & \mapsto & x \end{array} \right.$$

is a continuous function.

Problem 6

Evaluate the following two integrals:

(i) $\displaystyle\int_E x_1 \, \mathrm{d}x$ where $E = \{x \in \mathbb{R}^3 : 4 \leq x_1^2 + x_2^2 \leq 9, 0 \leq x_3 \leq 5 + x_1 + x_2\}$;

(ii) $\displaystyle\int_E \frac{\sin x_1}{x_1} \, \mathrm{d}x$ where $E \subset \mathbb{R}^2$ is the triangle with vertices $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} \pi \\ 0 \end{bmatrix}$, and $\begin{bmatrix} \pi \\ 1 \end{bmatrix}$.

Problem 7

Consider the following two statements about any Jordan set $F \subset \mathbb{R}^d$:

(i) $\mu(F) > 0$;

(ii) There exists a non-empty open set $U \subset \mathbb{R}^d$ with $U \subset F$.

Turn (i)$\heartsuit$(ii) into a true logical statement by replacing $\heartsuit$ with either $\Leftarrow$, $\Rightarrow$, or $\Leftrightarrow$. If your choice is $\Leftarrow$ (resp. $\Rightarrow$) rather than $\Leftrightarrow$, give an example for which $\Rightarrow$ (resp. $\Leftarrow$) is false.

Problem 8

Show that there exist uniquely determined numbers $\alpha, \beta \in \mathbb{R}$ that make the integral

$$I(\alpha, \beta) := \int_0^1 \big(\sin(\pi x) - \alpha - \beta x\big)^2 \mathrm{d}x$$

minimal. Find these minimising α, β.

Problem 9

Is the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \sin(\pi x^2)$ uniformly continuous? Justify your answer.