Math 217 Fall 2013 Midterm Solutions

NAME: ID#:

- There are four questions. Last two pages are scrap paper.
- Please write clearly and show enough work.

Question 1. (5 pts) *Let* $A := \{(x, y) \in \mathbb{R}^2 | \, x \geqslant 0, y \geqslant 0, x\, y \geqslant 1\}$. *Prove that* A *is convex.*

Solution. Take any $(x_1, y_1), (x_2, y_2) \in A$ and $t \in (0, 1)$. We prove that

$$(x, y) = t\,(x_1, y_1) + (1 - t)\,(x_2, y_2) \in A \tag{1}$$

It is clear that

$$x = t\,x_1 + (1 - t)\,x_2 \geqslant 0, \qquad y = t\,y_1 + (1 - t)\,y_2 \geqslant 0. \tag{2}$$

Now check

$$\begin{aligned} x\,y &= (t\,x_1 + (1 - t)\,x_2)\,(t\,y_1 + (1 - t)\,y_2) \\ &= t^2\,x_1\,y_1 + t\,(1 - t)\,(x_1\,y_2 + x_2\,y_1) + (1 - t)^2\,x_2\,y_2 \\ &\geqslant t^2 + t\,(1 - t)\left(\frac{x_1}{x_2} + \frac{x_2}{x_1}\right) + (1 - t)^2. \end{aligned} \tag{3}$$

Since

$$\frac{x_1}{x_2} + \frac{x_2}{x_1} = \frac{x_1^2 + x_2^2}{x_1\,x_2} \geqslant 2 \tag{4}$$

we have

$$x\,y \geqslant t^2 + 2\,t\,(1 - t) + (1 - t)^2 = 1. \tag{5}$$

Thus $(x, y) \in A$ and the proof ends.

Question 2. (5 pts) *Let* $A, B \subseteq \mathbb{R}^N$ *be open. Prove that* $C := \{\boldsymbol{x} + \boldsymbol{y} | \, \boldsymbol{x} \in A, \boldsymbol{y} \in B\}$ *is open.*

Solution. Take any $\boldsymbol{z}_0 \in C$. Then $\boldsymbol{z}_0 = \boldsymbol{x}_0 + \boldsymbol{y}_0$ for some $\boldsymbol{x}_0 \in A$, $\boldsymbol{y}_0 \in B$. Now since A is open, there is $r > 0$ such that $B(\boldsymbol{x}_0, r) \subseteq A$. Consequently $B(\boldsymbol{x}_0 + \boldsymbol{y}_0, r) \subseteq C$. So C is open.

Question 3. (5 pts) *Let* $\boldsymbol{f}\colon \mathbb{R}^2 \mapsto \mathbb{R}^2$ *be defined as*

$$\boldsymbol{f}(x, y) = \begin{pmatrix} x^2 - y^2 \\ 2\,x\,y \end{pmatrix}. \tag{6}$$

a) **(3 pts)** *Prove that for every nonzero* $(x_0, y_0) \in \mathbb{R}^2$*, there is an open set* $U \ni (x_0, y_0)$ *such that* $\boldsymbol{f}$ *is one-to-one on* U*.*

b) **(2 pts)** *Let* $U \ni (0, 0)$*. Is* $\boldsymbol{f}$ *one-to-one on* U*? Justify.*

Solution.

a) We calculate

$$\frac{\partial \boldsymbol{f}}{\partial(x, y)} = \begin{pmatrix} 2\,x & -2\,y \\ 2\,y & 2\,x \end{pmatrix} \tag{7}$$

whose determinant is $4\,(x^2 + y^2)$. At $(x_0,\, y_0) \neq (0,\, 0)$ the determinant is nonzero. Furthermore all partial derivatives are continuous so Implicit Function Theorem applies and the conclusion follows.

b) We show that for any $r > 0$, $\boldsymbol{f}$ is not one-to-one on $B(\mathbf{0}, r)$. To see this notice that $\boldsymbol{f}(x, y) = \boldsymbol{f}(-x, -y)$ for all (x, y).

Question 4. (10 pts) *Let $f\colon \mathbb{R}^2 \mapsto \mathbb{R}$ be defined as*

$$f(x,y) = \frac{x^3}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \text{ and } f(x,y) = 0 \text{ if } (x,y) = (0,0). \tag{8}$$

a) **(3 pts)** *Prove that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are bounded on $\mathbb{R}^2$.*

b) **(3 pts)** *Prove that $\frac{\partial f}{\partial \boldsymbol{v}}(0,0)$ exists for every nonzero $\boldsymbol{v} \in \mathbb{R}^2$.*

c) **(2 pts)** *Let $\boldsymbol{g}\colon \mathbb{R} \mapsto \mathbb{R}^2$ be differentiable with $\boldsymbol{g}(0) = (0,0)$ and $\boldsymbol{g}'(0) \neq (0,0)$. Define $F(t) := f(\boldsymbol{g}(t))$. Prove that $F(t)$ is differentiable at $t = 0$.*

d) **(2 pts)** *Prove that f is not differentiable at $(0,0)$.*

Solution.

a) When $(x,y) \neq (0,0)$ we calculate

$$\frac{\partial f}{\partial x} = \frac{3\,x^2}{x^2+y^2} - \frac{2\,x^4}{(x^2+y^2)^2}; \tag{9}$$

$$\frac{\partial f}{\partial y} = -\frac{2\,x^3\,y}{(x^2+y^2)^2}. \tag{10}$$

Since

$$x^2 \leqslant x^2+y^2, \qquad x^4 \leqslant (x^2+y^2)^2, \qquad |x^3\,y| = |x^2|\,|x\,y| \leqslant (x^2+y^2)^2 \tag{11}$$

we have

$$\left|\frac{\partial f}{\partial x}\right| \leqslant 5, \qquad \left|\frac{\partial f}{\partial y}\right| \leqslant 2 \tag{12}$$

for all $(x,y) \neq (0,0)$.

At $(0,0)$ we have $f(x,0) = x$, $f(0,y) = 0$. So $\frac{\partial f}{\partial x}(0,0) = 1$, $\frac{\partial f}{\partial y}(0,0) = 0$. Summarizing, we see that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are bounded on $\mathbb{R}^2$.

b) Take any $\boldsymbol{v} = (u,v) \neq (0,0)$. We calculate

$$f(0+t\,u, 0+t\,v) = \frac{(t\,u)^3}{(t\,u)^2+(t\,v)^2} = \frac{t\,u^3}{u^2+v^2}. \tag{13}$$

Thus clearly the directional derivative exists:

$$\frac{\partial f}{\partial \boldsymbol{v}} = \frac{u^3}{u^2+v^2}. \tag{14}$$

c) Denote $\boldsymbol{g}(t) := (u(t), v(t))$ and $\boldsymbol{g}'(0) := (a,b)$. Then we have

$$F(t) := f(\boldsymbol{g}(t)) = \frac{u(t)^3}{u(t)^2+v(t)^2}. \tag{15}$$

This leads to

$$\frac{F(t)-F(0)}{t} = \frac{u(t)^2}{u(t)^2+v(t)^2} \cdot \frac{u(t)}{t}. \tag{16}$$

It is clear that

$$\lim_{t \longrightarrow 0} \frac{u(t)}{t} = \lim_{t \longrightarrow 0} \frac{u(t)-u(0)}{t} = u'(0) = a. \tag{17}$$

On the other hand, we have

$$\lim_{t \longrightarrow 0} \frac{u(t)^2}{u(t)^2 + v(t)^2} = \lim_{t \longrightarrow 0} \frac{\left[\frac{u(t) - u(0)}{t}\right]^2}{\left[\frac{u(t) - u(0)}{t}\right]^2 + \left[\frac{v(t) - v(0)}{t}\right]^2} = \frac{a^2}{a^2 + b^2}. \tag{18}$$

Note that here we have used $a^2 + b^2 \neq 0$ which comes from the assumption that $\boldsymbol{g}'(0) \neq \mathbf{0}$. Therefore $F(t)$ is differentiable at $t = 0$.

d) We prove by contradiction. Since $\frac{\partial f}{\partial x}(0,0) = 1$, $\frac{\partial f}{\partial y}(0,0) = 0$, if f is differentiable at $(0,0)$ then $(Df)(0,0)(x, y) = x$, and

$$0 = \lim_{(x,y) \longrightarrow (0,0)} \frac{\left|\frac{x^3}{x^2 + y^2} - x\right|}{(x^2 + y^2)^{1/2}} = \lim_{(x,y) \longrightarrow (0,0)} \frac{|x\, y^2|}{(x^2 + y^2)^{3/2}}. \tag{19}$$

However

$$\frac{|x\, y^2|}{(x^2 + y^2)^{3/2}} = 2^{-3/2} \tag{20}$$

whenever $x = y$, independent of $(x^2 + y^2)^{1/2}$. Therefore $\lim_{(x,y) \longrightarrow (0,0)} \frac{|x\, y^2|}{(x^2 + y^2)^{3/2}} = 0$ does not hold and f is not differentiable at $(0,0)$.