Math 217 Fall 2013 Homework 9 Solutions

by Due Thursday Nov. 21, 2013 5pm

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. *Consider the following set:*

$$A := \left\{ \left(\frac{p}{q}, \frac{r}{q} \right) \mid p, q \in \mathbb{N}, (p, q), (r, q) \text{ co-prime} \right\} \cap [0, 1]^2. \tag{1}$$

Prove that for every $\alpha \in [0, 1]$, $\mu(A \cap \{x = \alpha\}) = \mu(A \cap \{y = \alpha\}) = 0$, *therefore*

$$\int_0^1 \mu(A \cap \{x = t\})\, \mathrm{d}t = \int_0^1 \mu(A \cap \{y = t\})\, \mathrm{d}t = 0, \tag{2}$$

but $\mu(A)$ *does not exist. (This example is constructed by A. Pringsheim in 1898).*

Solution.

Note that if $\alpha \notin \mathbb{Q}$, then $A \cap \{x = \alpha\} = A \cap \{y = \alpha\} = \varnothing$ so the (1-dimensional) measure is 0. If $\alpha \in \mathbb{Q}$, then $\alpha = \frac{m}{n}$ for some (m, n) co-prime. But then both $A \cap \{x = \alpha\}$ and $A \cap \{y = \alpha\}$ are finite set so again the measure is 0.

Since $A \subseteq \mathbb{Q} \times \mathbb{Q}$, it is clear that $A^o = \varnothing$ so $\mu_{\text{in}}(A) = 0$. To show that A is not measurable, we prove $\bar{A} = [0, 1]^2$. For any $(x, y) \in [0, 1]^2$ and any $r > 0$, there is always $n \in \mathbb{N}$ and k, l odd such that

$$\left| x - \frac{k}{2^n} \right| < \frac{r}{2}, \qquad \left| y - \frac{l}{2^n} \right| < \frac{r}{2}. \tag{3}$$

Thus

$$B((x, y), r) \cap A \neq \varnothing \tag{4}$$

and consequently $(x, y) \in \bar{A}$. Thus ends the proof.

Question 2. *Calculate*

$$\int_D x^2\, y^2\, \mathrm{d}(x, y) \tag{5}$$

where D *is the triangle enclosed by* $y = \frac{b}{a} x, y = 0, x = a$.

Solution. We have

$$\int_D x^2\, y^2\, \mathrm{d}(x,y) \;=\; \int_0^a \left[\int_0^{\frac{bx}{a}} x^2\, y^2\, \mathrm{d}y\right] \mathrm{d}x$$
$$= \frac{1}{3}\,\frac{b^3}{a^3}\int_0^a x^5\, \mathrm{d}x = \frac{a^3\, b^3}{18}. \qquad (6)$$

Question 3. *Calculate*

$$\int_D (x^2+y^2)\, \mathrm{d}(x,y) \qquad (7)$$

where D is enclosed by

$$y=a+x,\, y=x,\, y=a,\, y=3\,a. \qquad (8)$$

Solution. We have

$$\int_D (x^2+y^2)\, \mathrm{d}(x,y) = \int_a^{3a} \left[\int_{y-a}^{y} (x^2+y^2)\, \mathrm{d}x\right] \mathrm{d}y = 14\, a^4. \qquad (9)$$

Question 4. *Calculate the area enclosed by $y=x^2$ and $y^2=x$.*

Solution. Let D be the set. We have

$$\int_D 1\, \mathrm{d}(x,y) = \int_0^1 \left[\int_{x^2}^{\sqrt{x}} \mathrm{d}y\right] \mathrm{d}x = \int_0^1 (\sqrt{x}-x^2)\, \mathrm{d}x = \frac{1}{3}. \qquad (10)$$

Question 5. *Let $f(x,y)$ be continuous on $I:=[a,b]\times[c,d]$. Define for $(x,y)\in I$,*

$$F(x,y) := \int_{[a,x]\times[c,y]} f(u,v)\, \mathrm{d}(u,v). \qquad (11)$$

Prove that

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial^2 F(x,y)}{\partial y \partial x} = f(x,y). \qquad (12)$$

Solution. Since $f(x,y)$ is continuous, by Fubini we have

$$F(x,y) = \int_a^x \left[\int_c^y f(u,v)\, \mathrm{d}v\right] \mathrm{d}u. \qquad (13)$$

Now for fixed y, we show

$$\Phi(u) := \int_c^y f(u,v)\, \mathrm{d}v \qquad (14)$$

is a continuous function of u. Fix u_0. We show that $\Phi(u)$ is continuous at u_0.

Since f is continuous on I, it is uniformly continuous. Thus for every $\varepsilon > 0$, there is $r > 0$ such that for all $\|(u_1, v_1) - (u_2, v_2)\| < r$, $|f(u_1, v_1) - f(u_2, v_2)| < \frac{\varepsilon}{d - c}$. Now for every $|u - u_0| < r$, we have

$$|\Phi(u) - \Phi(u_0)| = \left| \int_c^y |f(u, v) - f(u_0, v)| \mathrm{d}v \right| < \varepsilon. \tag{15}$$

Therefore $\Phi(u)$ is continuous.

Now by FTC (single variable),

$$\frac{\partial F(x, y)}{\partial x} = \frac{\partial}{\partial x}\left[\int_a^x \Phi(u)\, \mathrm{d}u \right] = \Phi(x) = \int_c^y f(x, v)\, \mathrm{d}v. \tag{16}$$

Now since $f(x, v)$ is clearly continuous in v, applying FTC again we have

$$\frac{\partial^2 F(x, y)}{\partial y \partial x} = f(x, y). \tag{17}$$

The proof for $\frac{\partial^2 F(x, y)}{\partial x \partial y} = f(x, y)$ is similar.

Question 6. *Let $I := [a, b] \times [c, d]$. Let $f(x): [a, b] \mapsto \mathbb{R}$, $g(x): [c, d] \mapsto \mathbb{R}$. Let $F(x, y) := f(x)\, g(y)$.*

a) *Prove that $F(x, y)$ is integrable on I if f, g are integrable on $[a, b]$, $[c, d]$ respectively. Furthermore we have*

$$\int_I F(x, y)\, \mathrm{d}(x, y) = \left[\int_a^b f(x)\, \mathrm{d}x \right] \left[\int_c^d g(x)\, \mathrm{d}x \right]. \tag{18}$$

b) *Does it hold that $F(x, y)$ is integrable only if f, g are integrable?*

c) *Prove*

$$\left[\int_a^b f(x)\, \mathrm{d}x \right]^2 \leqslant (b - a) \int_a^b f(x)^2\, \mathrm{d}x \tag{19}$$

though studying

$$\int_{[a,b]^2} [f(x) - f(y)]^2\, \mathrm{d}(x, y). \tag{20}$$

Solution.

a) Note that it suffices to prove for $f, g \geqslant 0$. For general f, g we use

$$f(x)\, g(y) = f^+(x)\, g^+(y) - f^-(x)\, g^+(y) - f^+(x)\, g^-(y) + f^-(x)\, g^-(y) \tag{21}$$

where

$$f^+(x) := \max(f(x), 0),\ f^-(x) := \min(f(x), 0) \tag{22}$$

and g^+, g^- are defined similarly.

Let $n \in \mathbb{N}$, $h_1 := \frac{b-a}{n}$, $h_2 := \frac{d-c}{n}$. Define

$$I_{i,n} := [a + (i-1)\, h_1, a + i\, h_1], \quad J_{j,n} := [c + (j-1)\, h_2, c + j\, h_2], \qquad I_{ij,n} := I_{i,h} \times J_{j,h}. \tag{23}$$

For each i we set

$$f_{i,n} := \sup_{x \in I_{i,n}} f(x); \qquad g_{i,n} := \sup_{x \in J_{i,n}} g(x). \tag{24}$$

Then define

$$G_h(x, y) := f_{i,n}\, g_{j,n} \tag{25}$$

for all $(x, y) \in I_{ij,n}$. Now we have $G_h \geqslant F(x, y)$ and is a simple function. Thus

$$U(F, I) \leqslant \int_I G_h(x, y) = \left(\sum_{i=1}^{n} f_{i,n}\, h_1 \right) \left(\sum_{j=1}^{n} g_{j,n}\, h_2 \right). \tag{26}$$

Taking $n \longrightarrow \infty$ we have

$$U(F, I) \leqslant \left[\int_a^b f(x)\, \mathrm{d}x \right] \left[\int_c^d g(x)\, \mathrm{d}x \right] \tag{27}$$

by the integrability of f, g. Similarly we have

$$L(F, I) \geqslant \left[\int_a^b f(x)\, \mathrm{d}x \right] \left[\int_c^d g(x)\, \mathrm{d}x \right]. \tag{28}$$

Thus $U(F, I) = L(F, I) = \left[\int_a^b f(x)\, \mathrm{d}x \right] \left[\int_c^d g(x)\, \mathrm{d}x \right]$ and the conclusion follows.

b) "Only if' does not hold. For example take $f(x) = D(x)$ the Dirichlet function and $g(x) = 0$.

c) By a) $f(x)\, f(y)$ is integrable on $[a, b]^2$. Furthermore $f(x)$ integrable on $[a, b]$ means $f(x)^2$ is integrable on $[a, b]$. Now by a) we have $f(x)^2 = f(x)^2 \cdot 1$ is integrable on $[a, b]^2$. Similarly $f(y)^2$ is integrable on $[a, b]^2$.

We have, again through application of a),

$$\begin{aligned} 0 &\leqslant \int_{[a,b]^2} [f(x) - f(y)]^2\, \mathrm{d}(x, y) \\ &= \int_{[a,b]^2} f(x)^2\, \mathrm{d}(x, y) + \int_{[a,b]^2} f(y)^2\, \mathrm{d}(x, y) - 2 \int_{[a,b]^2} f(x)\, f(y)\, \mathrm{d}(x, y) \\ &= 2\, (b-a) \int_a^b f(x)^2\, \mathrm{d}x - 2 \left[\int_a^b f(x)\, \mathrm{d}x \right]^2 \end{aligned} \tag{29}$$

and the conclusion follows.

Remark. For the "only if" part, in fact the only problem is that one of f, g can be 0. But formulating a positive statement seems messy.