Math 217 Fall 2013 Homework 6

Due Thursday Oct. 31, 2013 5pm

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. *Let* $A \subseteq \mathbb{R}^N$ *be convex. Let* $r > 0$ *and* $B := \{\boldsymbol{x} \in \mathbb{R}^N |\ \mathrm{dist}(\boldsymbol{x}, A) < r\}$*, where* $\mathrm{dist}(\boldsymbol{x}, A) := \inf_{\boldsymbol{y} \in A} \|\boldsymbol{x} - \boldsymbol{y}\|$*. Prove that* B *is convex.*

Question 2. *Consider* $\boldsymbol{f}: \mathbb{R}^2 \mapsto \mathbb{R}^2$ *defined through*

$$\boldsymbol{f}(x, y) := \begin{pmatrix} x^3 - 3\,x\,y^2 \\ 3\,x^2\,y - y^3 \end{pmatrix}. \tag{1}$$

Prove:

a) *For every* $(x_0, y_0) \neq (0, 0)$ *there is open set* $U \ni (x_0, y_0)$ *such that* $\boldsymbol{f}$ *is one-to-one on* U*;*

b) *Let* U *be open and* $(0, 0) \in U$*. Then* $\boldsymbol{f}$ *is not one-to-one on* U*.*

Question 3. *Let* $f: (a, b) \mapsto \mathbb{R}$ *such that* $f^{(n+1)}(x)$ *is continuous. For any* $x_0, x \in (a, b)$*, write*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x, x_0). \tag{2}$$

a) *Prove that* $\frac{\partial R_n}{\partial x_0}$ *exists;*

b) *Calculate* $\frac{\partial R_n}{\partial x_0}$*;*

c) *Prove that*

$$R_n(x, x_0) = \frac{1}{n!} \int_{x_0}^{x} (x - y)^n f^{(n+1)}(y)\,\mathrm{d}y. \tag{3}$$

Question 4. *Calculate all third order partial derivartives for* $f(x, y) = x^4 + y^4 + 4\,x^2\,y^2$.

Question 5. *Let* $f(x, y, z) := x\,y\,z\,e^{x+y+z}$*. Find* $\frac{\partial^{p+q+r} f}{\partial x^p \partial y^q \partial z^r}$*. Here* $p, q, r \in \mathbb{N} \cup \{0\}$.

Question 6. *Let* $f: \mathbb{R}^2 \mapsto \mathbb{R}$*. Assume* $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ *are differentiable at* (x_0, y_0)*. Prove*

$$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0). \tag{4}$$