Math 217 Fall 2013 Homework 5 Solutions

Due Thursday Oct. 17, 2013 5pm

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. *Consider* $\boldsymbol{f}: \mathbb{R} \mapsto \mathbb{R}^3$ *defined through*

$$\boldsymbol{f}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}. \tag{1}$$

Find $t_1 < t_2$ *such that there is no* $\xi \in (t_1, t_2)$ *satisfying*

$$\boldsymbol{f}(t_2) - \boldsymbol{f}(t_1) = \boldsymbol{f}'(\xi)\,(t_2 - t_1). \tag{2}$$

Explain why this is not contradicting Mean Value Theorem.

Solution. Take $t_1 = 0, t_2 = 2\pi$. Then

$$\boldsymbol{f}(t_2) - \boldsymbol{f}(t_1) = \begin{pmatrix} 0 \\ 0 \\ 2\pi \end{pmatrix}. \tag{3}$$

On the other hand,

$$\boldsymbol{f}'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}. \tag{4}$$

If there is $\xi \in (0, 2\pi)$ such that

$$\begin{pmatrix} 0 \\ 0 \\ 2\pi \end{pmatrix} = \begin{pmatrix} -\sin \xi \\ \cos \xi \\ 1 \end{pmatrix} 2\pi \tag{5}$$

then necessarily

$$\cos \xi = \sin \xi = 0 \tag{6}$$

which contradicts the identity

$$(\cos \xi)^2 + (\sin \xi)^2 = 1. \tag{7}$$

Question 2. *Find* $f(x, y)$ *such that* f *is differentiable (meaning differentiable everywhere) but* $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ *are not continuous.*

Solution. Take $f(x, y) = \begin{cases} x^2 y^2 \sin \frac{1}{x} \sin \frac{1}{y} & x \neq 0, y \neq 0 \\ 0 & x = 0 \text{ or } y = 0 \end{cases}$.

- We first show that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exists at all (x_0, y_0).

  At $x_0 \neq 0, y_0 \neq 0$, direct calculation gives

$$\frac{\partial f}{\partial x} = 2\,x\,y^2 \sin\frac{1}{x} \sin\frac{1}{y} - y^2 \cos\frac{1}{x} \sin\frac{1}{y} \tag{8}$$

$$\frac{\partial f}{\partial y} = 2\,x^2\,y \sin\frac{1}{x} \sin\frac{1}{y} - x^2 \sin\frac{1}{x} \cos\frac{1}{y}. \tag{9}$$

At $x_0 = 0, y_0 \neq 0$, we have clearly $\frac{\partial f}{\partial y} = 0$. To calculate $\frac{\partial f}{\partial x}$ write

$$f(x, y_0) = \begin{cases} x^2\, y_0^2 \sin\frac{1}{x} \sin\frac{1}{y_0} & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{10}$$

which gives

$$\frac{\partial f}{\partial x}(0, y_0) = 0. \tag{11}$$

Similarly at $x_0 \neq 0, y_0 = 0$ we have $\frac{\partial f}{\partial x}(x_0, 0) = \frac{\partial f}{\partial y}(x_0, 0) = 0$.

Finally clearly $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$.

- Now we show that they are not continuous.

  Summarizing the above, we have

$$\frac{\partial f}{\partial x} = \begin{cases} 2\,x\,y^2 \sin\frac{1}{x} \sin\frac{1}{y} - y^2 \sin\frac{1}{x} \sin\frac{1}{y} & x \neq 0, y \neq 0 \\ 0 & x = 0 \text{ or } y = 0 \end{cases}. \tag{12}$$

Now we show that $\frac{\partial f}{\partial x}$ is not continuous at any $(0,\ y_0)$ with $y_0 \neq 0$. This is clear since $\lim_{x \longrightarrow 0} \frac{\partial f}{\partial x}(x, y_9)$ does not exist.

Similarly it can be shown that $\frac{\partial f}{\partial y}$ is not continuous at any $(x_0, 0)$ with $x_0 \neq 0$.

- Finally we prove that f is differentiable.

  Since $f(x,y) = \left(x^2 \sin\frac{1}{x}\right)\left(y^2 \sin\frac{1}{y}\right)$, it suffices to show that both $x^2 \sin\frac{1}{x}$ and $y^2 \sin\frac{1}{x}$ are differentiable as functions from $\mathbb{R}^2$ to $\mathbb{R}$ at every (x_0, y_0). We prove a more general statement:

> If $f(x)\colon \mathbb{R} \mapsto \mathbb{R}$ is differentiable, then $F\colon \mathbb{R}^N \mapsto \mathbb{R}$ defined through $F(\boldsymbol{x}) := f(x_1)$ is differentiable too.

Take any $\boldsymbol{x}_0 \in \mathbb{R}$. Denote $\boldsymbol{x}_0 = \begin{pmatrix} x_{01} \\ \vdots \\ x_{0N} \end{pmatrix}$. We have

$$\begin{aligned} \frac{|F(\boldsymbol{x}) - F(\boldsymbol{x}_0) - f'(x_{01})(x_1 - x_{01})|}{\|\boldsymbol{x} - \boldsymbol{x}_0\|} &= \frac{|f(x_1) - f(x_{01}) - f'(x_{01})(x_1 - x_{01})|}{\|\boldsymbol{x} - \boldsymbol{x}_0\|} \\ &\leqslant \frac{|f(x_1) - f(x_{01}) - f'(x_{01})(x_1 - x_{01})|}{|x_1 - x_{01}|}. \end{aligned} \tag{13}$$

This gives

$$\lim_{\boldsymbol{x} \longrightarrow \boldsymbol{x}_0} \frac{|F(\boldsymbol{x}) - F(\boldsymbol{x}_0) - f'(x_{01})(x_1 - x_{01})|}{\|\boldsymbol{x} - \boldsymbol{x}_0\|} = 0 \tag{14}$$

and the differentiability of F.

Remark. A better example may be

$$f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y} & x \neq 0, y \neq 0 \\ x^2 \sin\frac{1}{x} & y = 0 \\ y^2 \sin\frac{1}{y} & x = 0 \\ 0 & x = y = 0 \end{cases}. \tag{15}$$

Question 3. *Let* $f(x,y)=\sin(2x)\sin y\sin(2x+y)$ *and* $A:=\{(x,y)|\,x\geqslant 0, y\geqslant 0, 2x+y\leqslant\pi\}$*. Find* $\max_{(x,y)\in E} f(x,y)$*.*

Solution. First notice that $f(x,y)=0$ on ∂A while $f(x,y)>0$ in A^o. Therefore the maximizer must be in A^o and should satisfy the equations $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$. This gives

$$0=\frac{\partial f}{\partial x} = 2\cos(2x)\sin y\sin(2x+y)+2\sin(2x)\sin y\cos(2x+y) \quad (16)$$

$$0=\frac{\partial f}{\partial y} = \sin(2x)\cos y\sin(2x+y)+\sin(2x)\sin y\cos(2x+y) \quad (17)$$

These simplify to

$$\sin(4x+y)\sin y = 0 \quad (18)$$

$$\sin(2x)\sin(2x+2y) = 0. \quad (19)$$

Now as we are considering $x>0, y>0, 2x+y<\pi$, there must hold

$$4x+y = \pi \quad (20)$$

$$2x+2y = \pi \quad (21)$$

which leads to $x=\pi/6, y=\pi/3$. Since this is the only candidate and

$$f(\pi/6,\pi/3)=\frac{3\sqrt{3}}{8}>0, \quad (22)$$

the maximum is $\frac{3\sqrt{3}}{8}$.

Question 4. *Let* $z=Z(x,y)$ *be determined through the equation*

$$x\,y+y\,z+z\,x=1. \quad (23)$$

Find $\frac{\partial Z}{\partial x}, \frac{\partial Z}{\partial y}$ *without solving* Z *explicitly.*

Solution. Differentiating

$$x\,y+y\,Z(x,y)+Z(x,y)\,x=1 \quad (24)$$

we obtain

$$y+y\frac{\partial Z}{\partial x}+Z+x\frac{\partial Z}{\partial x} = 0 \Longrightarrow \frac{\partial Z}{\partial x}=-\frac{y+z}{x+y}; \quad (25)$$

$$x+y\frac{\partial Z}{\partial y}+Z+x\frac{\partial Z}{\partial y} = 0 \Longrightarrow \frac{\partial Z}{\partial y}=-\frac{x+z}{x+y}. \quad (26)$$

Question 5. *Let* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_N\in\mathbb{R}^N$ *be such that* $\|\boldsymbol{v}_i\|=1$ *for all* i, $\boldsymbol{v}_i\cdot\boldsymbol{v}_j=0$ *for all* $i\neq j$*. Let* $f\colon \mathbb{R}^N\mapsto\mathbb{R}$ *be differentiable. Prove*

$$\left(\frac{\partial f}{\partial \boldsymbol{v}_1}\right)^2+\cdots+\left(\frac{\partial f}{\partial \boldsymbol{v}_N}\right)^2=\left(\frac{\partial f}{\partial x_1}\right)^2+\cdots+\left(\frac{\partial f}{\partial x_N}\right)^2. \quad (27)$$

Solution. Since f is differentiable, we have

$$\frac{\partial f}{\partial \boldsymbol{v}_i}=(\mathrm{grad}\,f)^T\boldsymbol{v}_i=\boldsymbol{v}_i^T(\mathrm{grad}\,f). \quad (28)$$

This means

$$\left(\frac{\partial f}{\partial \boldsymbol{v}_i}\right)^2=(\mathrm{grad}\,f)^T\left(\boldsymbol{v}_i\,\boldsymbol{v}_i^T\right)(\mathrm{grad}\,f). \quad (29)$$

Now we have

$$\left(\frac{\partial f}{\partial \boldsymbol{v}_1}\right)^2 + \cdots + \left(\frac{\partial f}{\partial \boldsymbol{v}_N}\right)^2 = (\text{grad } f)^T \left(\sum_{i=1}^N \left(\boldsymbol{v}_i \boldsymbol{v}_i^T\right)\right) (\text{grad } f). \tag{30}$$

and all we need to show is

$$A := \sum_{i=1}^N \left(\boldsymbol{v}_i \boldsymbol{v}_i^T\right) = I. \tag{31}$$

We check

$$A\, \boldsymbol{v}_k = \sum_{i=1}^N \left(\boldsymbol{v}_i^T \boldsymbol{v}_k\right) \boldsymbol{v}_i = \boldsymbol{v}_k \tag{32}$$

for every $k = 1, 2, 3, \ldots, N$. Now denote $V \in \mathbb{R}^{N \times N}$ by

$$V = \left(\begin{array}{ccc} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_N \end{array}\right) \tag{33}$$

we have

$$A\, V = V. \tag{34}$$

Now we show that $\boldsymbol{v}_1, \ldots \boldsymbol{v}_N$ are linearly independent. Once this is done we know V is invertible and can conclude

$$A = A\, (V V^{-1}) = (A\, V)\, V^{-1} = V V^{-1} = I. \tag{35}$$

To see the linear independence, let $c_1, \ldots, c_N \in \mathbb{R}$ be such that

$$c_1\, \boldsymbol{v}_1 + \cdots + c_N\, \boldsymbol{v}_N = \mathbf{0}. \tag{36}$$

Now we have

$$c_1 = \boldsymbol{v}_1 \cdot [c_1\, \boldsymbol{v}_1 + \cdots + c_N\, \boldsymbol{v}_N] = 0. \tag{37}$$

Similarly we have $c_2 = \cdots = c_N = 0$ and therefore $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$ are linearly independent.

Question 6. *Let $\boldsymbol{f}: \mathbb{R}^N \mapsto \mathbb{R}^N$ have continuous partial derivatives. Let $\alpha > 0$. Assume that $\boldsymbol{f}$ satisfies*

$$\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| \geqslant \alpha\, \|\boldsymbol{x} - \boldsymbol{y}\| \tag{38}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$. Prove

a) $\det\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right) \neq 0$ *for all $\boldsymbol{x}$;*

b) For any fixed $\boldsymbol{y}_0 \in \mathbb{R}^N$, $F(\boldsymbol{x}) := \|\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x})\|$ reaches minimum but not maximum;

c) $\boldsymbol{f}(\mathbb{R}^N) = \mathbb{R}^N$.

Solution.

a) Assume the contrary. There is $\boldsymbol{x}_0 \in \mathbb{R}^N$ such that $\det\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right) = 0$. Then from linear algebra we know that there is $\boldsymbol{v} \in \mathbb{R}^N$ with $\|\boldsymbol{v}\| = 1$ such that

$$\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right) \boldsymbol{v} = \mathbf{0}. \tag{39}$$

As $\boldsymbol{f}$ has continuous partial derivatives, $\boldsymbol{f}$ is differentiable and

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{v}}(\boldsymbol{x}_0) = \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right) \boldsymbol{v} = \mathbf{0}. \tag{40}$$

In other words,

$$\lim_{t\longrightarrow 0} \frac{\|\boldsymbol{f}(\boldsymbol{x}_0 + t\,\boldsymbol{v}) - \boldsymbol{f}(\boldsymbol{x}_0)\|}{|t|} = 0. \tag{41}$$

Now take $\delta > 0$ such that

$$\frac{\|\boldsymbol{f}(\boldsymbol{x}_0 + t\,\boldsymbol{v}) - \boldsymbol{f}(\boldsymbol{x}_0)\|}{|t|} < \alpha \tag{42}$$

for all $0 < |t| < \delta$. Consider $\boldsymbol{x} = \boldsymbol{x}_0$, $\boldsymbol{y} = \boldsymbol{x}_0 + \frac{\delta}{2}\,\boldsymbol{v}$. We have

$$\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| < \alpha\,|t| = \alpha\,\|\boldsymbol{x} - \boldsymbol{y}\|, \tag{43}$$

contradiction.

b) Fix $\boldsymbol{y}_0$. Take any $\boldsymbol{x}_0 \in \mathbb{R}^N$. Denote $m{:}=\|\boldsymbol{y}_0 - \boldsymbol{f}(\boldsymbol{x}_0)\|$. Now for any $M > 0$, take $R > 0$ satisfying $R > M + m + \|\boldsymbol{x}_0\|$, then for any $\|\boldsymbol{x}\| > R$, we have

$$\begin{aligned} \|\boldsymbol{y}_0 - \boldsymbol{f}(\boldsymbol{x})\| &\geqslant \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0)\| - \|\boldsymbol{y}_0 - \boldsymbol{f}(\boldsymbol{x}_0)\| \\ &\geqslant \|\boldsymbol{x} - \boldsymbol{x}_0\| - m \\ &> R - \|\boldsymbol{x}_0\| - m > M. \end{aligned} \tag{44}$$

Thus we see clearly that $\lim_{\|\boldsymbol{x}\|\longrightarrow\infty} \|\boldsymbol{f}(\boldsymbol{x})\| = \infty$ and therefore the maximum does not exist.

Now take $M = m$. We see that for all $\boldsymbol{x} \in \overline{B(\boldsymbol{0}, R)}^c$, $\|\boldsymbol{y}_0 - \boldsymbol{f}(\boldsymbol{x})\| > m$. Consider the function $\|\boldsymbol{y}_0 - \boldsymbol{f}(\boldsymbol{x})\|$ on $\overline{B(\boldsymbol{0}, R)}$. It is continuous and therefore there the minimum is achieved. Since $\boldsymbol{x}_0 \in B(\boldsymbol{0}, R)$ we know that this minimum $\leqslant m$ and has to be the global minimum.

c) Take any $\boldsymbol{y}_0 \in \mathbb{R}^N$. All we need to prove is $\min_{\boldsymbol{x}\in\mathbb{R}^N} \|\boldsymbol{y}_0 - \boldsymbol{f}(\boldsymbol{x})\| = 0$. Assume the contrary. Take $\boldsymbol{x}_0$ to be the minimizer that is

$$\forall \boldsymbol{x} \in \mathbb{R}^N, \qquad 0 < \|\boldsymbol{y}_0 - \boldsymbol{f}(\boldsymbol{x}_0)\| \leqslant \|\boldsymbol{y}_0 - \boldsymbol{f}(\boldsymbol{x})\|. \tag{45}$$

Now set $\boldsymbol{v} := \frac{\boldsymbol{y}_0 - \boldsymbol{f}(\boldsymbol{x}_0)}{\|\boldsymbol{y}_0 - \boldsymbol{f}(\boldsymbol{x}_0)\|}$. Now as $\det\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right)(\boldsymbol{x}_0) \neq 0$, there is $r > 0$ such that an inverse function $\boldsymbol{g}$ exists: $\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{y})) = \boldsymbol{y}$. Therefore $B(\boldsymbol{y}_0, r) \subseteq \boldsymbol{f}(\mathbb{R}^N)$. In particular there is $\boldsymbol{x} \in \mathbb{R}^N$ such that

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}_0) + \frac{r}{2}\,\boldsymbol{v}. \tag{46}$$

Now we have

$$\|\boldsymbol{y}_0 - \boldsymbol{f}(\boldsymbol{x})\| < \|\boldsymbol{y}_0 - \boldsymbol{f}(\boldsymbol{x}_0)\|. \tag{47}$$

Contradiction.

Remark. Alternatively we can prove c) without proving b).

Since $\det\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right) \neq 0$, by the inverse function theorem, for any $\boldsymbol{y}_0 = \boldsymbol{f}(\boldsymbol{x}_0)$, there is $r > 0$ such that an inverse function $\boldsymbol{g}$ exists: $\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{y})) = \boldsymbol{y}$. Therefore $B(\boldsymbol{y}_0, r) \subseteq \boldsymbol{f}(\mathbb{R}^N)$. This means $\boldsymbol{f}(\mathbb{R}^N)$ is open.

One the other hand, if $\boldsymbol{y}_0 \in (\boldsymbol{f}(\mathbb{R}^N))^c$, and for any $r > 0$ there is $\boldsymbol{y} \in B(\boldsymbol{y}_0, r) \cap \boldsymbol{f}(\mathbb{R}^N)$, then we can find $\boldsymbol{x}_n$ such that $\boldsymbol{y}_n = \boldsymbol{f}(\boldsymbol{x}_n) \longrightarrow \boldsymbol{y}_0$. This gives

$$\|\boldsymbol{x}_n - \boldsymbol{x}_m\| \leqslant \alpha^{-1}\,\|\boldsymbol{y}_n - \boldsymbol{y}_m\|. \tag{48}$$

Since $\{\boldsymbol{y}_n\}$ is Cauchy, so is $\{\boldsymbol{x}_n\}$ and therefore there is $\boldsymbol{x}_0$ such that $\lim_{n\to\infty}\boldsymbol{x}_n = \boldsymbol{x}_0$.

Now as $\boldsymbol{f}$ has continuous partial derivatives, it is differentiable and therefore continuous. Consequently

$$\boldsymbol{y}_0 = \lim_{n\to\infty} \boldsymbol{f}(\boldsymbol{x}_n) = \boldsymbol{f}\Big(\lim_{n\to\infty} \boldsymbol{x}_n\Big) = \boldsymbol{f}(\boldsymbol{x}_0) \in \boldsymbol{f}(\mathbb{R}^N). \tag{49}$$

Contradiction. Therefore $\boldsymbol{f}(\mathbb{R}^N)$ is closed.

As $\boldsymbol{f}(\mathbb{R}^N)$ is both open and closed, it is either $\varnothing$ or $\mathbb{R}^N$. Clearly it is non-empty and therefore equals $\mathbb{R}^N$.