Math 217 Fall 2013 Homework 3

Due Thursday Oct. 3, 2013 5pm

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. (Convexity)

a) *Let* $E \subset \mathbb{R}^N$ *be defined by*

$$E := \{\boldsymbol{x} \in \mathbb{R}^N |\ \|\boldsymbol{x}\| < 1\} \cup \{(1, 0, \ldots, 0)\}. \tag{1}$$

Is E convex? Justify your answer.

b) *Let* $S \subseteq S(\mathbf{0}, 1) := \{\boldsymbol{x} \in \mathbb{R}^N |\ \|\boldsymbol{x}\| = 1\}$ *be any subset of the unit sphere. Define*

$$E := \{\boldsymbol{x} \in \mathbb{R}^N |\ \|\boldsymbol{x}\| < 1\} \cup S. \tag{2}$$

Is E convex? Justify your answer.

Question 2. (Limit) *Let* $k, l, m, n \in \mathbb{N}$. *Consider the following function:*

$$f(x, y) = \frac{x^k y^l}{x^{2m} + y^{2n}}. \tag{3}$$

Find all k, l, m, n *such that the limit* $\lim_{(x,y) \longrightarrow (0,0)} f(x, y)$ *exist. Justify your answer. (You may find the following Young's inequality useful:* $p, q > 0, \frac{1}{p} + \frac{1}{q} = 1 \Longrightarrow |x\, y| \leqslant \frac{|x|^p}{p} + \frac{|y|^q}{q}$*.)*

Question 3. (Limit at infinity) *Let* $\boldsymbol{f}: \mathbb{R}^N \mapsto \mathbb{R}^M$. *We define its limit at infinity as follows.* $\lim_{\boldsymbol{x} \longrightarrow \infty} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{L} \in \mathbb{R}^M$ *if and only if*

$$\forall \varepsilon > 0\ \exists R > 0\ \forall \boldsymbol{x} \text{ satisfying } \|\boldsymbol{x}\| > R \qquad \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{L}\| < \varepsilon. \tag{4}$$

Study the limit

$$\lim_{(x,y) \longrightarrow \infty} x\, y\, e^{-x^2 y^2}. \tag{5}$$

Does it exist? If it does, what is the limit? Justify your answer.

Question 4. (Continuity) *Let* $\boldsymbol{f}: \mathbb{R}^N \mapsto \mathbb{R}^M$ *be a linear function. Prove that it is continuous (that is, it is continuous at every point in its domain.)*

Question 5. (Open/closed sets) *Let* $A := \{(x, y) \in \mathbb{R}^2 |\ x < y\}$.

a) *Is it open? Is it closed?*

b) *Find its interior.*

c) *Find its closure.*

d) *Find its boundary.*

e) *Find its cluster points.*

Justify all your answers.

Question 6. (Open/closed sets) *Let* $A \subseteq \mathbb{R}^N$. *Prove* $(\overline{A^c})^c = A^o$.