Math 217 Fall 2013 Homework 2

Due Thursday Sept. 26, 2013 5pm

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. *The following are several possible strategies to prove Cauchy-Schwarz:*

$$|\boldsymbol{x}\cdot\boldsymbol{y}| = |x_1 y_1 + \cdots + x_N y_N| \leqslant (x_1^2 + \cdots + x_N^2)^{1/2} (y_1^2 + \cdots + y_N^2)^{1/2} = \|\boldsymbol{x}\| \, \|\boldsymbol{y}\|. \quad (1)$$

Pick any one (or come up with your own) idea and write down a detailed proof.

- *Approach 1.*
  Mathematical induction.
- *Approach 2.*
  Let $t \in \mathbb{R}$. *Then* $(\boldsymbol{x} - t\,\boldsymbol{y}) \cdot (\boldsymbol{x} - t\,\boldsymbol{y}) \geqslant 0$ *for all* t. *Write the left hand side as a quadratic polynomial of* t.
- *Approach 3.*
  Use $x_i y_i = \left(\frac{x_i}{k}\right)(y_i k) \leqslant \frac{1}{2}(x_i^2 k^{-2} + y_i^2 k^2)$. *Choose appropriate* k.

Question 2. *Let* $E \subseteq \mathbb{R}^N$. *Define its distance function* $d\colon \mathbb{R}^N \mapsto \mathbb{R}$ *as*

$$d(\boldsymbol{x}) := \inf_{\boldsymbol{y}\in E} \operatorname{dist}(\boldsymbol{x}, \boldsymbol{y}) = \inf_{\boldsymbol{y}\in E} \|\boldsymbol{x} - \boldsymbol{y}\|. \quad (2)$$

Prove that $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$, $|d(\boldsymbol{x}) - d(\boldsymbol{y})| \leqslant \|\boldsymbol{x} - \boldsymbol{y}\|$.

Question 3.

a) *Prove that the following are both norms on* $\mathbb{R}^N$:

$$\|\boldsymbol{x}\|_\infty := \max_{i=1,\ldots,N} \{|x_i|\}; \qquad \|\boldsymbol{x}\|_1 := |x_1| + |x_2| + \cdots + |x_N|; \quad (3)$$

b) *Let* X *be a linear vector space with norm* $\|\cdot\|$. *Prove the following: If one can define an inner product* $(\cdot,\cdot)$ *such that* $\|x\| = (x,x)^{1/2}$, *then for any* $x, y \in X$,

$$\|x + y\|^2 + \|x - y\|^2 = 2\,(\|x\|^2 + \|y\|^2). \quad (4)$$

c) *Find a norm on* $\mathbb{R}^N$ *that cannot be defined through an inner product. Justify your answer.*

Question 4. *Let* $O \in \mathbb{R}^{N\times N}$ *be such that* $\|O\,\boldsymbol{x}\| = \|\boldsymbol{x}\|$ *for any* $\boldsymbol{x} \in \mathbb{R}^N$. *Prove that* O *is orthogonal. Please prove it directly and do not use any theorem from linear algebra.*

Question 5. *Let* $D = \mathrm{diag}(d_1, \ldots, d_N)$ *be a diagonal matrix with all the* d_i*'s distinct. Let* $A \in \mathbb{R}^{N \times N}$ *be such that* $A\,D = D\,A$*. What can we conclude about* A*? Justify your answer.*

Question 6. (Twin Prime Conjecture) *Earlier this year, Prof. Yitang Zhang of University of New Hampshire made history through proving the following result:*

$$\liminf_{n \longrightarrow \infty} (p_{n+1} - p_n) < 7 \times 10^7 \tag{5}$$

where p_n *is the* n*-th prime number.*

a) *Prove that the Twin Prime Conjecture "There are infinitely many pairs of prime numbers with difference 2" is equivalent to*

$$\liminf_{n \longrightarrow \infty} (p_{n+1} - p_n) = 2. \tag{6}$$

b) *One step of his proof is basically the following. Assume*

$$\sum_{d < D^2, d | \mathcal{P}} \sum_{c \in \mathcal{C}_i(d)} |\Delta(\theta, d, c)| \leqslant x\, (\log x)^{-A}, \tag{7}$$

for some $A > 0$ *and*

$$\sum_{c \in \mathcal{C}_i(d)} |\Delta(\theta, d, c)| \leqslant x\, (\log x)/d; \qquad \sum_{d < D^2, d | \mathcal{P}} \tau_3(d)^2\, \rho_2(d)^2\, d^{-1} \leqslant (\log x)^B \tag{8}$$

for some $B > 0$*. Then we have*

$$\mathcal{E} := \left| \sum_{d < D^2, d | \mathcal{P}} \tau_3(d)\, \rho_2(d) \sum_{c \in \mathcal{C}_i(d)} |\Delta(\theta, d, c)| \right| \leqslant x\, (\log x)^{\frac{B+1-A}{2}}. \tag{9}$$

for any $A > 0$*. Prove the above claim using Cauchy-Schwarz.*