Math 217 Fall 2013 Homework 10 Solutions

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- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. *Let $L\colon \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a linear transformation with matrix representation $A := \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ where $c \in \mathbb{R}$.*

a) *Find the matrix representation for L^{-1}.*

b) *Let $I := [a_1, a_2] \times [b_1, b_2] \subseteq \mathbb{R}^2$. Prove that $L^{-1}(I)$ is Jordan measurable and $\mu(L^{-1}(I)) = \mu(I)$. (Hint: Fubini).*

c) *Let $B \subseteq \mathbb{R}^2$ be a simple graph. Prove that $L^{-1}(B)$ is Jordan measurable and $\mu(L^{-1}(B)) = \mu(B)$.*

d) *Let $E \subseteq \mathbb{R}^2$ be Jordan measurable. Prove that $L^{-1}(E)$ is Jordan measurable and $\mu(L^{-1}(E)) = \mu(E)$.*

e) *Let $E \subseteq \mathbb{R}^2$ be Jordan measurable and let $f(x, y)$ be Riemann integrable on E. Prove that $\tilde{f}(u, v) := f(L(u, v))$ is Riemann integrable on $L^{-1}(E)$ and furthermore*

$$\int_E f(x, y)\, \mathrm{d}(x, y) = \int_{L^{-1}(E)} \tilde{f}(u, v)\, \mathrm{d}(u, v). \tag{1}$$

Solution.

a) The matrix representation is $A^{-1} = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}$.

b) We have

$$L^{-1}(I) := \{(x, y) |\, a_1 \leqslant x + c\, y \leqslant a_2, b_1 \leqslant y \leqslant b_2\}. \tag{2}$$

Clearly $\mu(\partial L^{-1}(I)) = 0$ so $L^{-1}(I)$ is Jordan measurable. Furthermore it is clear that for each fixed y_0, the slice $L^{-1}(I) \cap \{y = y_0\}$ is also Jordan measurable.

To calculate its measure, we have

$$\begin{aligned} \mu(L^{-1}(I)) &:= \int_{L^{-1}(I)} \mathrm{d}(x, y) \\ &= \int_{b_1}^{b_2} \left[\int_{a_1 - cy}^{a_2 - cy} \mathrm{d}x \right] \mathrm{d}y \\ &= (a_2 - a_1)\, (b_2 - b_1) = \mu(I). \end{aligned} \tag{3}$$

c) As B is a simple graph, $B = \cup_{i=1}^n I_i$ where $i \neq j \Longrightarrow I_i^o \cap I_j^o = \varnothing$. Thus we have

$$\mu(L^{-1}(B)) = \mu(\cup_{i=1}^n L^{-1}(I_i)) \leqslant \sum_{i=1}^n \mu(L^{-1}(I_i)) = \sum_{i=1}^n \mu(I_i) = \mu(B). \tag{4}$$

On the other hand, for any $a \in (0,\, 1)$ we can find compact intervals $J_i \subseteq I_i^o$ such that $i \neq j \Longrightarrow J_i \cap J_j = \varnothing$ and $\mu(\cup_{i=1}^n J_i) > a\, \mu(B)$. Since L^{-1} is one-to-one, we have

$$\mu(L^{-1}(B)) \geqslant \mu(L^{-1}(\cup_{i=1}^n J_i)) = \mu(\cup_{i=1}^n L^{-1}(J_i)) = \sum_{i=1}^n \mu(L^{-1}(J_i)) = \sum_{i=1}^n \mu(J_i) > a\, \mu(B). \tag{5}$$

Since a is arbitrary, we have $\mu(L^{-1}(B)) = \mu(B)$.

d) Now for any $E \subseteq \mathbb{R}^2$ measurable and for any $a \in (0,1)$, we can find simple graphs $B \subseteq E \subseteq C$ such that $\mu(B) > a\, \mu(E) > a^2\, \mu(C)$. Now clearly $L^{-1}(B) \subseteq L^{-1}(E) \subseteq L^{-1}(C)$. Consequently

$$\mu(B) = \mu(L^{-1}(B)) \leqslant \mu_{\text{in}}(L^{-1}(E)) \leqslant \mu_{\text{out}}(L^{-1}(E)) \leqslant \mu(L^{-1}(C)) = \mu(C). \tag{6}$$

By our choices of B, C we have

$$a\, \mu(E) < \mu_{\text{in}}(L^{-1}(E)) \leqslant \mu_{\text{out}}(L^{-1}(E)) < a^{-1}\, \mu(E). \tag{7}$$

The arbitrariness of a now gives $\mu_{\text{in}}(L^{-1}(E)) = \mu_{\text{out}}(L^{-1}(E)) = \mu(E)$ and the conclusions of d) follow.

e) For any $\varepsilon > 0$, take simple functions $h \leqslant f \leqslant g$ such that $\int_E h(x, y)\, \mathrm{d}(x, y) + \varepsilon > \int_E f(x, y)\, \mathrm{d}(x, y) > \int_E g(x, y)\, \mathrm{d}(x, y) - \varepsilon$.

Now assume $h(x, y) = \sum_{i=1}^n c_i\, 1_{A_i}(x, y)$. Then

$$\begin{aligned}
\int_E h(x, y)\, \mathrm{d}(x, y) &= \sum_{i=1}^n c_i \int_E 1_{A_i}(x, y)\, \mathrm{d}(x, y) \\
&= \sum_{i=1}^n c_i \mu(A_i \cap E) \\
&= \sum_{i=1}^n c_i\, \mu(L^{-1}(A_i \cap E)) \\
&= \sum_{i=1}^n c_i\, \mu(L^{-1}(A_i) \cap L^{-1}(E)) \\
&= \sum_{i=1}^n c_i \int_{L^{-1}(E)} 1_{L^{-1}(A_i)}(u, v)\, \mathrm{d}(u, v) \\
&= \sum_{i=1}^n c_i \int_{L^{-1}(E)} 1_{A_i}(L(u, v))\, \mathrm{d}(u, v) \\
&= \int_{L^{-1}(E)} \tilde{h}(u, v)\, \mathrm{d}(u, v)
\end{aligned} \tag{8}$$

where

$$\tilde{h}(u,v) := h(L(u,v)) \tag{9}$$

is still a simple function. Similarly we have

$$\int_E g(x,y)\,\mathrm{d}(x,y) = \int_{L^{-1}(E)} \tilde{g}(u,v)\,\mathrm{d}(u,v) \tag{10}$$

with $\tilde{g}(u,v) = g(L(u,v))$ a simple function.

Now observe that $\tilde{h}(u,v) \leqslant \tilde{f}(u,v) \leqslant \tilde{g}(u,v)$. The conclusion follows from

$$\int_{L^{-1}(E)} [\tilde{g}(u,v) - \tilde{h}(u,v)]\,\mathrm{d}(u,v) = \int_E [g(x,y) - h(x,y)]\,\mathrm{d}(x,y) < 2\,\varepsilon \tag{11}$$

and the arbitrariness of ε.

Question 2. *Let A be enclosed by $x + y = \pm 1$ and $x - y = \pm 1$. Calculate*

$$\int_A \sin(x+y)\,\mathrm{d}(x,y) \tag{12}$$

a) *using Fubini directly;*

b) *using change of variables and then Fubini.*

Solution.

a) We have

$$A = \{(x,y) |\; -1 \leqslant x + y \leqslant 1, -1 \leqslant x - y \leqslant 1\}. \tag{13}$$

Therefore

$$\begin{aligned}
\int_A \sin(x+y)\,\mathrm{d}(x,y) &= \int_{-1}^{1} \left[\int_{|x|-1}^{1-|x|} \sin(x+y)\,\mathrm{d}y\right] \mathrm{d}x \\
&= \int_{-1}^{1} [-\cos(x+y)]_{y=|x|-1}^{y=1-|x|}\,\mathrm{d}x \\
&= \int_{-1}^{1} [\cos(x+|x|-1) - \cos(x+1-|x|)]\,\mathrm{d}x \\
&= \int_0^1 [\cos(2\,x-1) - \cos 1]\,\mathrm{d}x \\
&\quad + \int_{-1}^{0} [\cos(-1) - \cos(2\,x+1)]\,\mathrm{d}x \\
&= \frac{\sin(2\,x-1)}{2}\Big|_{x=0}^{x=1} - \frac{\sin(2\,x+1)}{2}\Big|_{x=-1}^{x=0} \\
&= \sin 1 - \sin 1 = 0.
\end{aligned} \tag{14}$$

b) We apply the change of variables:

$$u=x+y, v=x-y. \tag{15}$$

Then

$$x=\frac{u+v}{2}, y=\frac{u-v}{2} \Longrightarrow T(x,y)=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{16}$$

Therefore

$$|\det DT|=\frac{1}{2}. \tag{17}$$

Furthermore

$$T^{-1}(A)=[-1,1]^2. \tag{18}$$

So we have

$$\begin{aligned}\int_A \sin(x+y)\,\mathrm{d}(x,y) &= \int_{T^{-1}(A)} \sin(u)\,\frac{1}{2}\,\mathrm{d}(u,v) \\ &= \frac{1}{2}\int_{-1}^{1}\left[\int_{-1}^{1}\sin u\,\mathrm{d}u\right]\mathrm{d}v \\ &= \frac{1}{2}\int_{-1}^{1} 0\,\mathrm{d}v=0. \end{aligned} \tag{19}$$

Question 3. *Let $A\subseteq\mathbb{R}^3$ be the intersection of the ball $x^2+y^2+z^2\leqslant a^2$ and $x^2+y^2\leqslant a\,x$. Calculate its volume.*

Solution. We have

$$A:=\left\{(x,y,z)|\, x^2+y^2+z^2\leqslant a^2, \left(x-\frac{a}{2}\right)^2+y^2\leqslant\left(\frac{a}{2}\right)^2\right\}. \tag{20}$$

Notice that

$$\left\{(x,y)|\left(x-\frac{a}{2}\right)^2+y^2\leqslant\left(\frac{a}{2}\right)^2\right\}\subseteq\{(x,y)|\, x^2+y^2\leqslant a^2\}. \tag{21}$$

Therefore the volume is

$$V=\int_{(x-\frac{a}{2})^2+y^2\leqslant(\frac{a}{2})^2} 2\sqrt{a^2-x^2-y^2}\,\mathrm{d}(x,y). \tag{22}$$

Changing polar coordinates, we have

$$V=4\int_0^{\pi/2}\left[\int_0^{a\cos\theta}\sqrt{a^2-r^2}\,r\,\mathrm{d}r\right]\mathrm{d}\theta=\frac{4\,a^3}{3}\int_0^{\pi/2}(1-\sin^3\theta)\,\mathrm{d}\theta=\frac{4}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)a^3. \tag{23}$$

Question 4. *Calculate*

$$I = \int_{\left\{(x,y)\,|\,\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1\right\}} \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\,\mathrm{d}(x,y). \tag{24}$$

Solution. Make change of variables:

$$u = \frac{x}{a},\ \ v = \frac{y}{b} \Longrightarrow x = a\,u,\, y = b\,v \Longrightarrow |\det(DT)| = a\,b. \tag{25}$$

Then we have

$$I = a\,b \int_{\{(u,v)\,|\,u^2+v^2\leqslant 1\}} \sqrt{u^2+v^2}\,\mathrm{d}(u,v). \tag{26}$$

Now apply polar coordinates:

$$\begin{aligned} I &= a\,b \int_0^{2\pi} \left[\int_0^1 r^2\,\mathrm{d}r\right] \mathrm{d}\theta \\ &= \frac{2\,a\,b\,\pi}{3}. \end{aligned} \tag{27}$$

Question 5. *Calculate*

$$I = \int_A (x^2+y^2+z^2)\,\mathrm{d}(x,y,z) \tag{28}$$

where

$$A := \left\{(x,y,z)\,|\; x^2+y^2+z^2 \leqslant 1, \sqrt{x^2+y^2} \leqslant z\right\}. \tag{29}$$

Solution. Using spherical coordinates, we have

$$T^{-1}(A) = \left\{(r,\varphi,\psi)\,|\; 0 \leqslant r \leqslant 1, 0 \leqslant \varphi \leqslant 2\,\pi, 0 \leqslant \psi \leqslant \frac{\pi}{4}\right\}. \tag{30}$$

Thus

$$\begin{aligned} I &= \int_{T^{-1}(A)} r^4 \sin\psi\,\mathrm{d}(r,\varphi,\psi) \\ &= \int_0^1 r^4 \left[\int_0^{2\pi} \left[\int_0^{\pi/4} \sin\psi\,\mathrm{d}\psi\right] \mathrm{d}\varphi\right] \mathrm{d}r \\ &= \frac{\pi}{5}\left(2-\sqrt{2}\right). \end{aligned} \tag{31}$$

Question 6. *Let Ω be a ball with radius 1 and center $(0,0,1)$. Assume its density function is*

$$\rho(x,y,z) = \frac{1}{x^2+y^2+z^2}. \tag{32}$$

Find its center of mass.

Solution. By symmetry it is clear that the center of mass is on the z axis. Denote it by $(0, 0, z_0)$. Thus we only need to calculate through spherical coordinates:

$$\begin{aligned} z_0 &= \frac{1}{M}\int_\Omega \frac{z}{x^2+y^2+z^2}\,\mathrm{d}(x,y,z) \\ &= \frac{1}{M}\int_0^{2\pi}\left[\int_0^{\pi/2} \sin\psi\cos\psi\left[\int_0^{2\sin\psi} r\,\mathrm{d}r\right]\mathrm{d}\psi\right]\mathrm{d}\varphi \\ &= \frac{1}{M}\int_0^{2\pi}\left[\int_0^{\pi/2} 2\,(\sin\psi)^3\,(\cos\psi)\,\mathrm{d}\psi\right]\mathrm{d}\varphi \\ &= \frac{4\,\pi}{M}\int_0^1 u^3\,\mathrm{d}u \\ &= \frac{\pi}{M}. \end{aligned} \tag{33}$$

On the other hand through similar (but simpler) calculation we have

$$\begin{aligned} M &= \int_\Omega \frac{1}{x^2+y^2+z^2}\,\mathrm{d}(x,y,z) \\ &= \int_0^{2\pi}\left[\int_0^{\pi/2} \cos\psi\left[\int_0^{2\sin\psi} \mathrm{d}r\right]\mathrm{d}\psi\right]\mathrm{d}\varphi \\ &= 2\,\pi. \end{aligned} \tag{34}$$

Therefore $z_0 = 1/2$ and the center of mass for Ω is $(0, 0, 1/2)$.