Math 217 Fall 2013 Homework 10

by Due Thursday Nov. 28, 2013 5pm

- This homework consists of 6 problems of 5 points each. The total is 30.
- You need to fully justify your answer – prove that your function indeed has the specified property – for each problem.
- Please read this week's lecture notes before working on the problems.

Question 1. *Let $L: \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a linear transformation with matrix representation $A := \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ where $c \in \mathbb{R}$.*

a) *Find the matrix representation for L^{-1}.*

b) *Let $I := [a_1, a_2] \times [b_1, b_2] \subseteq \mathbb{R}^2$. Prove that $L^{-1}(I)$ is Jordan measurable and $\mu(L^{-1}(I)) = \mu(I)$. (Hint: Fubini).*

c) *Let $B \subseteq \mathbb{R}^2$ be a simple graph. Prove that $L^{-1}(B)$ is Jordan measurable and $\mu(L^{-1}(B)) = \mu(B)$.*

d) *Let $E \subseteq \mathbb{R}^2$ be Jordan measurable. Prove that $L^{-1}(E)$ is Jordan measurable and $\mu(L^{-1}(E)) = \mu(E)$.*

e) *Let $E \subseteq \mathbb{R}^2$ be Jordan measurable and let $f(x, y)$ be Riemann integrable on E. Prove that $\tilde{f}(u, v) := f(L(u, v))$ is Riemann integrable on $L^{-1}(E)$ and furthermore*

$$\int_E f(x, y)\, \mathrm{d}(x, y) = \int_{L^{-1}(E)} \tilde{f}(u, v)\, \mathrm{d}(u, v). \tag{1}$$

Question 2. *Let A be enclosed by $x + y = \pm 1$ and $x - y = \pm 1$. Calculate*

$$\int_A \sin(x + y)\, \mathrm{d}(x, y) \tag{2}$$

a) *using Fubini directly;*

b) *using change of variables and then Fubini.*

Question 3. *Let $A \subseteq \mathbb{R}^3$ be the intersection of the ball $x^2 + y^2 + z^2 \leqslant a^2$ and $x^2 + y^2 \leqslant a\, x$. Calculate its volume.*

Question 4. *Calculate*

$$I = \int_{\left\{(x,y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1\right\}} \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}\, \mathrm{d}(x, y). \tag{3}$$

Question 5. *Calculate*

$$I=\int_A (x^2+y^2+z^2)\,\mathrm{d}(x,y,z) \tag{4}$$

where

$$A:=\left\{(x,y,z)|\; x^2+y^2+z^2\leqslant 1, \sqrt{x^2+y^2}\leqslant z\right\}. \tag{5}$$

Question 6. *Let* Ω *be a ball with radius* 1 *and center* $(0,0,1)$. *Assume its density function is*

$$\rho(x,y,z)=\frac{1}{x^2+y^2+z^2}. \tag{6}$$

Find its center of mass.