Real numbers

Definition of real numbers

Definition 1. *We call any cut ξ a "positive real number".*

Remark 2. If we assume the existence of $\mathbb{R}$, then a "cut" is simply a set of the form $\{x \in \mathbb{Q}^+ | \, x < r\}$ where $r \in \mathbb{R}^+$. Such "cheating"-type understanding is often useful in designing proofs.

Notation. *As in the following we will not use the idea of "cut" anymore, from now on we will simply denote real numbers by any convenient letters (English or Greek).*

Definition 3. *Let the symbol 0 be defined through the property: For any positive real number x, $0 + x = x + 0 = x$.*

Exercise 1. Prove that 0 is different from any positive real number defined in the previous section.

Definition 4. *For any positive real number x, let the symbol $-x$ be defined through the property*

$$(-x) + x = x + (-x) = 0. \tag{1}$$

These numbers are called "negative numbers".

Exercise 2. Prove that $-(x + y) = (-x) + (-y)$.

Definition 5. *Positive real numbers, negative real numbers, and 0 together are called "real numbers".*

Definition 6. *We extend the operation "$-$" as follows.*

- *If $x > 0$, $-x$ is already defined;*
- *If $x = 0$, $-x := 0$;*
- *If $x < 0$, then there is ξ such that $x = -\xi$, define $-x := \xi$.*

Order

Definition 7. *Let x, y be real numbers. Then we say $x > y$ if and only if*

- *x, y positive $x > y$, or*
- *x, y negative, $-x < -y$, or*
- *$x = 0$, y negative, or*
- *$y = 0$, x positive, or*
- *x positive, y negative.*

We say $x < y$ if and only if $y > x$.

Exercise 3. For any $x, y \in \mathbb{R}$, exactly one of the following holds:

$$x = y, \quad x > y, \quad x < y. \tag{2}$$

Exercise 4. $x > 0$ if and only if it is a positive real number.

Definition 8. *Let x, y be real numbers. Then we say $x = y$ if and only if*

- *x, y positive, $x = y$, or*
- *$x = y = 0$, or*
- *x, y negative, $-x = -y$.*

Exercise 5. Prove that $=$ is an equivalence relation.

Definition 9. *Let x, y be real numbers. We say $x \geqslant y$ if and only if $x > y$ or $x = y$. We say $x \leqslant y$ if and only if $y \geqslant x$.*

Addition and subtraction

Definition 10. *Let x, y be real numbers. We define their sum $x + y$ as follows.*

- *If $x, y > 0$, already defined;*
- *If $x = 0$, $x + y = y$; If $y = 0$, $x + y = x$.*
- *If $x > 0, y < 0$, consider three sub-cases:*
    - *If $x > (-y)$, then define $x - y := u$ where u is the unique solution to $(-y) + u = x$.*
    - *If $x = \eta$, then by definition of negative numbers $x - y := 0$.*
    - *If $x < (-y)$, then define $x - y := -u$ where u is the unique solution to $x + u = (-y)$.*
- *If $x < 0$, $y > 0$ the definition is similar.*

Exercise 6. Prove that $x + y = y + x$.

Exercise 7. Define the absolute value function as

$$|x| := \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}. \tag{3}$$

Prove that $|-x| = |x|$.

Exercise 8. Prove $|x + y| \leqslant |x| + |y|$.

Definition 11. *Let x, y be real numbers. We define their difference $x - y := x + (-y)$.*

Exercise 9. Prove that $x - y = -(y - x)$.

Exercise 10. Prove that $x - y > 0 \Longleftrightarrow x > y$.

Exercise 11. Prove that for every $x \in \mathbb{R}$, there are $y, z \in \mathbb{R}^+$ such that $x = y - z$.

Exercise 12. Prove that $(x + y) + z = x + (y + z)$.

Multiplication and division

Definition 12. *Let $x, y \in \mathbb{R}$. We define their product $x \cdot y$ (or $x\, y$ when no confusion arises) through*

- *If $x, y > 0$, already defined;*
- *If $x = 0$ or $y = 0$, $x \cdot y := 0$;*
- *If $x, y < 0$, $x \cdot y := (-x) \cdot (-y)$;*
- *If $x > 0, y < 0$, $x \cdot y := -(x \cdot (-y))$;*
- *If $x < 0, y > 0$, $x \cdot y := -((-x) \cdot y)$.*

Exercise 13. If $x\, y = 0$, then at least one of x, y is 0.

Exercise 14. $|x\, y| = |x| \cdot |y|$.

Exercise 15. $x\, y = y\, x$.

Exercise 16. $x \cdot 1 = x$.

Exercise 17. $(-x)\,(-y) = x\, y$.

Exercise 18. $(x\, y)\, z = x\,(y\, z)$.

Exercise 19. $x\,(y + z) = x\, y + x\, z$.

Exercise 20. There is a unique u solving $x\, u = y$ if $x \neq 0$.

Theorem 13. $x, y \in \mathbb{R}$. *Then* $x\, y = (-x)\,(-y)$.

Proof. We have

$$(x + (-x))\, y = 0 \Longrightarrow x\, y + (-x)\, y = 0. \tag{4}$$

On the other hand

$$(-x)\,(y + (-y)) = 0 \Longrightarrow (-x)\, y + (-x)\,(-y) = 0. \tag{5}$$

Therefore $x\, y = (-x)\,(-y)$. □

Recall that for any $x \in \mathbb{R}^+$, there is a unique y such that $x\, y = 1$. From the above theorem we see that $(-y)$ is the unique solution to $(-x)\,(-y) = 1$. Thus division can be easily defined for all real numbers (denominator $\neq 0$).

Dedekind's fundamental theorem

Theorem 14. *Let $A \cup B = \mathbb{R}$ with*

i. A, B not empty;

ii. $A \cap B = \varnothing$;

iii. $\forall x \in A, y \in B,\ x < y$.

Then there is exactly one real number z *such that* $\{x|\ x < z\} \subseteq A, \{x|\ x > z\} \subseteq B$.

Proof.

- Uniqueness.

  If there are $z_1 \neq z_2$, then wlog $z_1 < z_2$. Then $\frac{z_1 + z_2}{2} \in A \cap B$. Contradiction.

- Existence.

  1. $A \cap \mathbb{R}^+ \neq \varnothing$. Consider $A \cap \mathbb{Q}$. If there is $x = \max(A \cap \mathbb{Q})$, set $\xi = A \cap \mathbb{Q}^+ - \{x\}$. Otherwise set $\xi = A \cap \mathbb{Q}^+$.

     Now it can be check that ξ is a cut and therefore determine a real number z.

     For any $x < z$, if $x < 0$, since $0 < z$ we have $x \in A$; If $x > 0$ we have $x < \frac{x + z}{2} < z$ and is therefore in A.

     Similarly we can show $\{x|\ x > z\} \subseteq B$.

  2. $A \cap \mathbb{R}^+ = \varnothing$, $0 \in A$. Then we prove that $z = 0$.

  3. Other cases are similarly discussed. $\square$