(Positive) Rational Numbers

Definition

Definition 1. *Consider all ordered pairs* (x_1, x_2) *with* $x_1, x_2 \in \mathbb{N}$. *Define the equivalance*

$$(x_1, x_2) \sim (y_1, y_2) \Longleftrightarrow x_1\, y_2 = x_2\, y_1. \tag{1}$$

Theorem 2. $\sim$ *is indeed an equivalence relation, that is*

1. $(x_1, x_2) \sim (x_1, x_2)$;
2. $(x_1, x_2) \sim (y_1, y_2) \Longleftrightarrow (y_1, y_2) \sim (x_1, x_2)$;
3. $(x_1, x_2) \sim (y_1, y_2)$ *and* $(y_1, y_2) \sim (z_1, z_2)$ *then* $(x_1, x_2) \sim (z_1, z_2)$.

Proof. Exercise. □

Exercise 1. $(x_1, x_2) \sim (x_1\, x, x_2\, x)$ for any $x \in \mathbb{N}$.

Notation. *From now on we will denote* (x_1, x_2) *by* $\frac{x_1}{x_2}$, *or* x_1/x_2.

Definition 3. $\frac{x_1}{x_2} > \frac{y_1}{y_2}$ *if and only if* $x_1\, y_2 > y_1\, x_2$. $\frac{x_1}{x_2} < \frac{y_1}{y_2}$ *if and only if* $\frac{y_1}{y_2} > \frac{x_1}{x_2}$.

Theorem 4. *Exactly one of the following three holds:*

$$\frac{x_1}{x_2} \sim \frac{y_1}{y_2}, \qquad \frac{x_1}{x_2} > \frac{y_1}{y_2}, \qquad \frac{x_1}{x_2} < \frac{y_1}{y_2}. \tag{2}$$

Exercise 2. If $\frac{x_1}{x_2} < \frac{y_1}{y_2}$ and $\frac{y_1}{y_2} < \frac{z_1}{z_2}$ then $\frac{x_1}{x_2} < \frac{z_1}{z_2}$.

Definition 5. $\frac{x_1}{x_2} \geqslant \frac{y_1}{y_2}$ *if and only if* $\frac{x_1}{x_2} > \frac{y_1}{y_2}$ *or* $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$. $\frac{x_1}{x_2} \leqslant \frac{y_1}{y_2}$ *if and only if* $\frac{x_1}{x_2} < \frac{y_1}{y_2}$ *or* $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$.

Exercise 3. If $\frac{x_1}{x_2} < \frac{y_1}{y_2}$ and $\frac{y_1}{y_2} \leqslant \frac{z_1}{z_2}$ then $\frac{x_1}{x_2} < \frac{z_1}{z_2}$.

Exercise 4. If $\frac{x_1}{x_2} \geqslant \frac{y_1}{y_2}$ and $\frac{y_1}{y_2} \geqslant \frac{x_1}{x_2}$, then $\frac{x_1}{x_2} = \frac{y_1}{y_2}$.

Theorem 6. *If* $\frac{x_1}{x_2} < \frac{y_1}{y_2}$ *then there is* $\frac{z_1}{z_2}$ *such that* $\frac{x_1}{x_2} < \frac{z_1}{z_2} < \frac{y_1}{y_2}$.

Proof. Set

$$z_1 = x_1 + y_1, \qquad z_2 = x_2 + y_2. \tag{3}$$

Then it satisfies the requirement. □

Problem 1. Define addition, subtraction, multiplication, division, and justify their properties, in particular those related to order.

Definition 7. *We define a positive rational number X to be an equivalence class of all pairs $\frac{y_1}{y_2}$ equivalent to a fixed pair $\frac{x_1}{x_2}$. We denote the set of positive rational numbers $\mathbb{Q}^+$.*

Definition 8. *Two rational numbers X, Y are equal, denoted $X = Y$, if there are $\frac{x_1}{x_2} \in X$ and $\frac{y_1}{y_2} \in Y$ such that $\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$.*

Remark 9. We can easily define $X < Y, X \leqslant Y, X > Y, X \geqslant Y$.

Theorem 10. *Let X, Y be rational numbers. Then there is an natural number z such that $z X > Y$.*

Proof. Take $\frac{x_1}{x_2} \in X, \frac{y_1}{y_2} \in Y$. We need to prove that there is $z \in \mathbb{N}$ such that $z x_1 y_2 > x_2 y_1$. More generally, we prove that for any $x, y \in \mathbb{N}$, there is z such that $z x > y$.

Fix x, Let $\mathfrak{M} := \{y \in \mathbb{N} | \exists z \in \mathbb{N}, z x > y\}$.

- $1 \in \mathfrak{M}$. Two cases.
  - $x = 1$. Then we have $1' \cdot x = x + x > x = 1$.
  - $x \neq 1$. Then there is $u \in \mathbb{N}$ such that $x = u' = 1 + u > 1$.

  Therefore $1 \in \mathfrak{M}$.

- $y \in \mathfrak{M}$ then $y' \in \mathfrak{M}$.

  Since $y \in \mathfrak{M}$ there is $z \in \mathbb{N}$ such that $z x > y$. Thus there is $u \in \mathbb{N}$ such that $z x = y + u$. Now consider $z' x = z x + x = y + u + x$. It is easy to see that $u + x \neq 1$ thus there is $v \in \mathbb{N}$ such that $u + x = v'$. This gives

$$z' x = y + u + x = y + v' = y' + v > y'. \tag{4}$$

Thus ends the proof. $\square$

Theorem 11. *Define $S := \left\{X \in \mathbb{Q}^+ | X \sim \frac{x}{1} \text{ for some } x \in \mathbb{N}\right\}$. Denote $\frac{1}{1}$ by 1. For any $X \in S$, define $X' := X + \frac{1}{1}$. Then S satisfies Axioms 1 – 5 of $\mathbb{N}$ and is therefore $\mathbb{N}$.*

Notation. *In the following we will stop using capital letters to denote rational numbers.*

Dedekind cuts

Definition 12. *A subset $\xi \subseteq \mathbb{Q}^+$ is called a "cut" if and only if*

i. it contains a rational number, but does not contain all rational numbers;

ii. every rational number in the set is smaller than every rational number not in the set;

iii. it does not contain a greatest rational number.

Exercise 5. Let $\xi \subseteq \mathbb{Q}$ be a cut. Prove that

$$[x \in \xi] \Longrightarrow [\forall y < x, y \in \xi]; \qquad [x \notin \xi] \Longrightarrow [\forall y > x, y \notin \xi]. \tag{5}$$

Exercise 6. Let $x \in \mathbb{Q}$. Then $\xi := \{y \in \mathbb{Q} | y < x\}$ is a cut.

Definition 13. *Two cuts ξ, η are said to be equal, denoted $\xi = \eta$, if they are equal as sets.*

Theorem 14. *"=" is an equivalence relation, that is*

$$\xi = \xi; \qquad \xi = \eta \Longrightarrow \eta = \xi; \qquad \xi = \eta, \eta = \zeta \Longrightarrow \xi = \zeta. \tag{6}$$

Definition 15. *Let ξ, η be two cuts. Say $\xi < \eta$ if and only if $\xi \subsetneq \eta$. Say $\xi > \eta$ if and only if $\eta < \xi$. Say $\xi \leqslant \eta$ if $\xi \subseteq \eta$ and $\xi \geqslant \eta$ if $\eta \leqslant \xi$.*

Exercise 7. For any ξ, η exactly one of the following is true: $\xi = \eta; \xi > \eta; \xi < \eta$.

Exercise 8. If $\xi \leqslant \eta$ and $\eta \leqslant \xi$, then $\xi = \eta$.

Addition of cuts

Theorem 16. *Let ξ, η be cuts. Then*

$$\zeta := \{x + y | \, x \in \xi, y \in \eta\} \tag{7}$$

is also a cut.

Proof. Since ξ, η are cuts they are not empty. Thus ζ is not empty. On the other hand, ξ, η are not $\mathbb{Q}$, therefore there is $a, b \in \mathbb{Q}$ such that $a \notin \xi, b \notin \eta$. By definition of cuts we have

$$\forall x \in \xi, x < a; \qquad \forall y \in \eta, y < b. \tag{8}$$

This gives

$$\forall z \in \zeta, z < a + b. \tag{9}$$

Therefore ζ is not $\mathbb{Q}$.

Next we prove that if $z \in \zeta$ then all $z' < z$ also belongs to ζ. Let $z \in \zeta$. Then there are $x \in \xi, y \in \eta$ such that $z = x + y$. Now for any $z' < z$, we have $\frac{z'}{z} x < x \Longrightarrow \frac{z'}{z} x \in \xi$ and similarly $\frac{z'}{z} y \in \eta$. Now we have

$$\frac{z'}{z} x + \frac{z'}{z} y = z'. \tag{10}$$

Now we prove that if $z \notin \zeta$ and $z' > z$, then $z' \notin \zeta$. This is obvious through proof by contradiction and what we have just proved.

Finally since neither ξ nor η has a greatest number, for any $z \in \zeta$ there is always $z' \in \zeta$ satisfying $z' > z$. $\square$

Definition 17. *Let ξ, η be cuts. Then define their sum to be*

$$\xi + \eta := \{x + y | \, x \in \xi, y \in \eta\}. \tag{11}$$

Exercise 9. Prove $\xi + \eta > \xi$.

Exercise 10. Prove $\xi + \eta = \eta + \xi$.

Exercise 11. Prove $(\xi + \eta) + \zeta = \xi + (\eta + \zeta)$.

Exercise 12. Let ξ be a cut. Let $a \in \mathbb{Q}^+$. Then there are $x \in \xi, y \notin \xi$ such that $y - x = a$. (Hint: Consider $x + n\, a$ for $n \in \mathbb{N}$)

Theorem 18. *If $\xi > \eta$, then there is a unique ζ such that $\eta + \zeta = \xi$.*

Proof. Uniqueness is trivial. We prove existence. Since $\xi > \eta$, the set

$$A := \{x \in \mathbb{Q}^+ | \, x \in \xi, x \notin \eta\} \tag{12}$$

is not empty. Define

$$\zeta := \{x - y | \, x, y \in A, x > y\}. \tag{13}$$

We first prove that it is a cut.

i. First we prove ζ is not empty.

   Sincer $\xi > \eta$, we have $\eta \subsetneq \xi$ and thus there is $y \in \xi, y \notin \eta$. Now as ξ is a cut, there is $x \in \xi$ such that $x > y$. Since y is a cut, $x > y \notin \eta \Longrightarrow x \notin \eta$. Thus we have two numbers $x, y \in A$ with $x > y$. By definition $x - y \in \zeta$.

ii. Next we prove $\zeta \neq \mathbb{Q}^+$.

   Since ξ is a cut, there is $z \in \mathbb{Q}^+$ such that $z \notin \xi$. Now for any $w \in \zeta$, there are x, $y \in \xi$ such that $w = x - y < x < z$. Thus $z \notin \zeta$.

iii. Then we prove $\forall x \in \zeta, y \notin \zeta, x < y$.

   Assume the contrary, that is $x \in \zeta$, $y \notin \zeta$ but $x > y$ ($x = y$ is ruled out by $y \notin \zeta$). Since $x \in \zeta$, there are $u, v \in A$ such that $x = u - v$. Thus we have

   $$u = x + v > y + v. \tag{14}$$

   As $u \in \xi$ is a cut, $y + v \in \xi$. One the other hand, $v \notin \eta$ which is a cut implies $y + v \notin \eta$ since $y + v > v$. Thus we have $y + v \in A$. But now

   $$y = (y + v) - v \in \zeta \tag{15}$$

   Contradiction.

iv. Finally we prove that ζ does not have a greatest element.

   Take any $y \in \zeta$. By definition there are $u, v \in A$ such that $y = u - v$. Now since ξ is a cut and $u \in \xi$, there is $w \in \xi$ such that $w > u$. Since η is a cut and $u \notin \eta$, $w \notin \eta$. Thus $w \in A$ and we have

   $$\zeta \ni x := w - v > u - v = y. \tag{16}$$

   Therefore y is not a greatest element of ζ. □

Definition 19. *Denote that ζ in the above theorem as $\xi - \eta$.*

Multiplication of cuts

Theorem 20. *Let ξ, η be cuts. Then*

$$\zeta := \{x \, y | \, x \in \xi, y \in \eta\} \tag{17}$$

is also a cut.

Proof. Left as exercise. □

Definition 21. *The ζ above is called the product of ξ, η, denoted $\xi \cdot \eta$ (or $\xi\eta$ if no confusion arises)*

Exercise 13. $\xi\eta = \eta\xi$;

Exercise 14. $(\xi\eta)\zeta = \xi(\eta\zeta)$;

Exercise 15. $\xi(\eta + \zeta) = \xi\eta + \xi\zeta$.

Exercise 16. If $\xi > \eta$, then $\xi\zeta > \eta\zeta$.

Theorem 22. *Let $\mathbf{1}$ be the cut $\{x \in \mathbb{Q}^+ | x < 1\}$. Then for any cut ξ, we have $\mathbf{1} \cdot \xi = \xi \cdot \mathbf{1} = \xi$.*

Proof. Exercise. □

Theorem 23. *For any ξ, there is a unique η such that $\xi\eta = \mathbf{1}$.*

Proof. Take

$$\eta := \left\{ \frac{y}{z} | \ y \in \mathbf{1}, z \notin \xi \right\}. \tag{18}$$

Then η is the cut we need. □

Exercise 17. Finish the above proof.

Exercise 18. ξ, η are cuts. Then there is a unique v such that $\xi v = \eta$.

Exercise 19. Define division.

Problem 2. Let $x \in \mathbb{Q}$. Define $\xi_x := \{y \in \mathbb{Q} | \ y < x\}$. Then it is a cut. Prove that such "rational cuts" satisfy the same arithmetic rules as rational numbers.

Problem 3. Let $n \in \mathbb{N}$. Defind $\xi_n := \{y \in \mathbb{Q} | \ y < n\}$. Then it is a cut. Prove that such "integral cuts" satisfy the axioms for natural numbers.

Problem 4. If $\xi < \eta$, then there is a rational number Z such that $\xi < Z < \eta$.

Exercise 20. For each ζ, the equation $\xi^2 := \xi \cdot \xi = \zeta$ has exactly one solution.

Definition 24. *Any cut which is not a rational number is called an irrational number.*

Theorem 25. *There exists an irrational number.*

Proof. It suffices to show that the solution to $\xi^2 = 1'$ (recall that this is the successor of 1) is irrational. Otherwise we would have $\xi = \frac{x}{y}$ where $x, y \in \mathbb{N}$. Take x, y such that y is smallest. We easily see that $y < x < 1' y$. Set $x - y = u$, then $u < y$. Next set $y - u = t$. Then

$$\begin{aligned} x\,x + t\,t &= (y + u)\,(y + u) + t\,t \\ &= (y\,y + 1'\,y\,u) + (u\,u + t\,t) \\ &= (y\,y + 1'\,u\,(u + t)) + (u\,u + t\,t) \\ &= (y\,y + 1'\,(u\,u)) + (u\,u + 1'\,u\,t + t\,t) \\ &= y\,y + 1'\,u\,u + y\,y \\ &= 1'\,y\,y + 1'\,u\,u \\ &= x\,x + 1'\,u\,u. \end{aligned} \tag{19}$$

Thus

$$\frac{t}{u} \cdot \frac{t}{u} = 1'. \tag{20}$$

But $u < y$. Contradiction. □

Exercise 21. Re-write the above proof using human (mathematical) language.