Applications to triple integrals

Cylindrical coordinates

Cylindrical coordinate transformation:

$$T(r,\theta,z):=\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}. \tag{1}$$

It is easy to calculate

$$|\det(DT)|=r. \tag{2}$$

Example 1. (PKU3) Calculate

$$\int_A z\,\mathrm{d}(x,y,z) \tag{3}$$

where A is bounded by $x^2+y^2+z^2=4$ and $x^2+y^2=3\,z$.

We apply cylindrical coordinate transformation T. Then

$$\begin{aligned}\int_A z\,\mathrm{d}(x,y,z) &= \int_0^{2\pi}\left[\int_0^{\sqrt{3}}\left[\int_{r^2/3}^{\sqrt{4-r^2}} r\,z\,\mathrm{d}z\right]\mathrm{d}r\right]\mathrm{d}\theta \\ &= 2\,\pi\int_0^{\sqrt{3}}\frac{r}{2}\left(4-r^2-\frac{r^4}{9}\right)\mathrm{d}r \\ &= \frac{13}{4}\,\pi. \end{aligned} \tag{4}$$

Alternatively, we can calculate

$$\begin{aligned}\int_A z\,\mathrm{d}(x,y,z) &= \int_0^1\left[\int_{x^2+y^2\leqslant 3z} z\,\mathrm{d}(x,y)\right]\mathrm{d}z+\int_1^2\left[\int_{x^2+y^2\leqslant 4-z^2} z\,\mathrm{d}(x,y)\right]\mathrm{d}z \\ &= \int_0^1 3\,\pi\,z^2\,\mathrm{d}z+\int_1^2 \pi\,z\,(4-z^2)\,\mathrm{d}z \\ &= \pi+\pi\left(6-\frac{15}{4}\right)=\frac{13}{4}\,\pi. \end{aligned} \tag{5}$$

Example 2. (PKU3) Calcualte

$$\int_A (x^2+y^2)^{1/2}\,\mathrm{d}(x,y,z) \tag{6}$$

where

$$A:=\left\{(x,y,z)|\ \sqrt{x^2+y^2}\leqslant z\leqslant 1\right\}. \tag{7}$$

Apply cylindrical coordinates we have

$$\int_A (x^2+y^2)^{1/2}\,\mathrm{d}(x,y,z)=\int_0^{2\pi}\left[\int_0^1\left[\int_0^z r^2\,\mathrm{d}r\right]\mathrm{d}z\right]\mathrm{d}\theta=\frac{\pi}{6}. \tag{8}$$

Alternatively,

$$\int_A (x^2+y^2)^{1/2}\,\mathrm{d}(x,y,z)=\int_0^1\left[\int_{x^2+y^2\leqslant z^2}(x^2+y^2)^{1/2}\,\mathrm{d}(x,y)\right]\mathrm{d}z=\frac{\pi}{6}. \tag{9}$$

Spherical coordinates

The transformation is

$$T(\rho,\varphi,\psi):=\begin{pmatrix}\rho\cos\varphi\cos\psi\\ \rho\sin\varphi\cos\psi\\ \rho\sin\psi\end{pmatrix} \tag{10}$$

with

$$|\det(DT)|=\rho^2\cos\psi. \tag{11}$$

Example 3. (PKU3) Calculate

$$\int_\Omega (x^2+y^2+z^2)\,\mathrm{d}(x,y,z) \tag{12}$$

where

$$\Omega=\{(x,y,z)|\,x^2+y^2+z^2\leqslant 2\,z\}. \tag{13}$$

We have

$$\int_\Omega (x^2+y^2+z^2)\,\mathrm{d}(x,y,z) = \int_0^{2\pi}\left[\int_0^{\pi/2}\left[\int_0^{2\sin\psi}\rho^4\cos\psi\,\mathrm{d}\rho\right]\mathrm{d}\psi\right]\mathrm{d}\varphi=\frac{32}{15}\,\pi. \tag{14}$$

Other change of variables

Example 4. (PKU3) Calculate

$$\int_\Omega (x^2+y^2+z^2)\,\mathrm{d}(x,y,z) \tag{15}$$

where

$$\Omega:=\left\{(x,y,z)|\,\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1\right\}. \tag{16}$$

We do the change of variable

$$T(\rho,\varphi,\psi):=\begin{pmatrix}a\,\rho\cos\varphi\cos\psi\\ b\,\rho\sin\varphi\cos\psi\\ c\,\rho\sin\psi\end{pmatrix}. \tag{17}$$

Then

$$|\det(DT)|=a\,b\,c\,\rho^2\cos\psi \tag{18}$$

and

$$T^{-1}(\Omega)=\left\{(\rho,\varphi,\psi)|\,0\leqslant\rho\leqslant 1, 0\leqslant\varphi\leqslant 2\,\pi, -\frac{\pi}{2}\leqslant\psi\leqslant\frac{\pi}{2}\right\} \tag{19}$$

Therefore

$$\begin{aligned}\int_\Omega (x^2+y^2+z^2)\,\mathrm{d}(x,y,z) &= \int_0^{2\pi}\left[\int_{-\pi/2}^{\pi/2}\left[\int_0^1 (a^2\cos^2\varphi\cos^2\psi+b^2\sin^2\varphi\cos^2\psi+c^2\sin^2\psi)\,a\,b\,c\,\rho^4\cos\psi\,\mathrm{d}\rho\right]\mathrm{d}\psi\right]\mathrm{d}\varphi\\ &= \frac{4}{15}\,\pi\,(a^2+b^2+c^2).\end{aligned} \tag{20}$$

Example 5. (PKU3) Let $h := \sqrt{\alpha^2+\beta^2+\gamma^2} > 0$ and let $f(x)$ be continuous on $[-h, h]$, prove

$$\int_{\Omega} f(\alpha\, x+\beta\, y+\gamma\, z)\,\mathrm{d}(x, y, z) = \pi \int_{-1}^{1} (1-w^2)\, f(h\, w)\,\mathrm{d}w. \tag{21}$$

Here $\Omega := \{(x, y, z) |\, x^2+y^2+z^2 \leqslant 1\}$.

Proof. Let $\boldsymbol{e}_1 := h^{-1}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$. Take $\boldsymbol{e}_2, \boldsymbol{e}_3 \in \mathbb{R}^3$ such that $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is an orthonormal basis. Let

$$O := (\ \boldsymbol{e}_1\ \ \boldsymbol{e}_2\ \ \boldsymbol{e}_3\ ) \in \mathbb{R}^{3\times 3}. \tag{22}$$

Define the transformation:

$$T(u, v, w) = u\, \boldsymbol{e}_1 + v\, \boldsymbol{e}_2 + w\, \boldsymbol{e}_3. \tag{23}$$

Then we have

$$|\det(DT)| = |\det O| = 1 \tag{24}$$

$$T^{-1}(\Omega) = \{(u, v, w) |\, u^2+v^2+w^2 \leqslant 1\} \tag{25}$$

and

$$u = \boldsymbol{x}\cdot \boldsymbol{e}_1, \quad v = \boldsymbol{x}\cdot \boldsymbol{e}_2, \quad w = \boldsymbol{x}\cdot \boldsymbol{e}_3 = h^{-1}(\alpha\, x+\beta\, y+\gamma\, z). \tag{26}$$

Now apply change of variable and Fubini, we have

$$\begin{aligned}\int_{\Omega} f(\alpha\, x+\beta\, y+\gamma\, z)\,\mathrm{d}(x, y, z) &= \int_{T^{-1}(\Omega)} f(h\, w)\,\mathrm{d}(u, v, w)\\ &= \int_{-1}^{1}\left[\int_{u^2+v^2 \leqslant 1-w^2} f(h\, w)\,\mathrm{d}(u, v)\right]\mathrm{d}w\\ &= \pi \int_{-1}^{1} (1-w^2)\, f(h\, w)\,\mathrm{d}w. \end{aligned}\tag{27}$$

The proof ends. □

Example 6. (PKU3) Calculate

$$\int_{\Omega} x\, y\, z\,\mathrm{d}(x, y, z) \tag{28}$$

where $\Omega \subseteq \{(x, y, z) |\, x \geqslant 0, y \geqslant 0, z \geqslant 0\}$ is enclosed by $z = \frac{x^2+y^2}{m}$, $z = \frac{x^2+y^2}{n}$, $x\, y = a^2$, $x\, y = b^2$, $y = \alpha\, x$, $y = \beta\, x$ $(0 < a < b, 0 < \alpha < \beta, 0 < m < n)$.

We make the change of variable:

$$u = \frac{z}{x^2+y^2}, v = x\, y, w = \frac{y}{x}. \tag{29}$$

The corresponding T is:

$$x = \sqrt{\frac{v}{w}}, \quad y = \sqrt{v\, w}, \quad z = u\, v\left(w + \frac{1}{w}\right). \tag{30}$$

We check that it is indeed a normal transformation.

Now

$$|\det(DT)| = \frac{v}{2\,w}\left(w + \frac{1}{w}\right) \tag{31}$$

and

$$T^{-1}(\Omega) = \left\{(u, v, w)|\ \frac{1}{n} \leqslant u \leqslant \frac{1}{m},\ \ a^2 \leqslant v \leqslant b^2,\ \ \alpha \leqslant w \leqslant \beta\right\}. \tag{32}$$

Now

$$\begin{aligned}\int_\Omega x\,y\,z\,\mathrm{d}(x, y, z) &= \int_{T^{-1}(\Omega)} \frac{1}{2}\,u\,v^3\left(w + \frac{2}{w} + \frac{1}{w^3}\right)\mathrm{d}(u, v, w)\\ &= \int_{1/n}^{1/m}\left[\int_{a^2}^{b^2}\left[\int_\alpha^\beta \frac{1}{2}\,u\,v^3\left(w + \frac{2}{w} + \frac{1}{w^3}\right)\mathrm{d}w\right]\mathrm{d}v\right]\mathrm{d}u\\ &= \frac{1}{32}\left(\frac{1}{m^2} - \frac{1}{n^2}\right)(b^8 - a^8)\left[(\beta^2 - \alpha^2)\left(1 + \frac{1}{\alpha^2\,\beta^2}\right) + 4\ln\frac{\beta}{\alpha}\right].\end{aligned} \tag{33}$$