Applications to double integrals

Polar coordinates

Example 1. (PKU3) Calculate

$$\int_A (x^2+y^2)^{1/2}\,\mathrm{d}(x,y) \tag{1}$$

where

$$A := \{(x,y)|\,(x-a)^2+y^2>a^2;\,(x-2\,a)^2+y^2<4\,a^2;\,y>x\}. \tag{2}$$

Change of variables:

$$T(r,\theta)=\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}, \qquad \det(DT(r,\theta))=\det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}=r \tag{3}$$

with

$$T^{-1}(A) := \left\{(r,\theta)|\,\theta\in\left[\frac{\pi}{4},\frac{\pi}{2}\right], r\in[2\,a\cos\theta, 4\,a\cos\theta]\right\}. \tag{4}$$

Application of change of variable and Fubini now gives

$$\begin{aligned}\int_A (x^2+y^2)^{1/2}\,\mathrm{d}(x,y) &= \int_{T^{-1}(A)} r^2\,\mathrm{d}(r,\theta) \\ &= \int_{\pi/4}^{\pi/2}\left[\int_{2a\cos\theta}^{4a\cos\theta} r^2\,\mathrm{d}r\right]\mathrm{d}\theta \\ &= \frac{56\,a^3}{3}\int_{\pi/4}^{\pi/2}\cos^3\theta\,\mathrm{d}\theta \\ &= \frac{112-70\sqrt{2}}{9}\,a^3. \end{aligned} \tag{5}$$

Example 2. Calculate

$$\int_A e^{-(x^2+y^2)}\,\mathrm{d}(x,y) \tag{6}$$

where $A := \{(x,y)|\,x^2+y^2\leqslant a^2\}$ for some $a>0$.

Change of variable:

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{7}$$

Thus

$$T(r,\theta)=\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}, \qquad \det(DT(r,\theta))=\det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}=r \tag{8}$$

and

$$T^{-1}(A)=[0,a]\times[0,2\,\pi]. \tag{9}$$

Thus

$$\int_A e^{-(x^2+y^2)}\,\mathrm{d}(x,y)=\int_{T^{-1}(A)} e^{-r^2}\,r\,\mathrm{d}(r,\theta)=\int_0^{2\pi}\left[\int_0^a e^{-r^2}\,r\,\mathrm{d}r\right]\mathrm{d}\theta=\pi\left(1-e^{-a^2}\right). \tag{10}$$

Note that we have applied Fubini.

Example 3. Let the Gamma function be defined as

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1} \, \mathrm{d}x. \tag{11}$$

Prove that $\Gamma(1/2) = \sqrt{\pi}$.

Proof. Since

$$\Gamma(1/2) = \int_0^\infty e^{-x} x^{-1/2} \, \mathrm{d}x, \tag{12}$$

we try the change of variable $x = x_1^2$. This gives

$$\Gamma(1/2) = \int_0^\infty e^{-x_1^2} x_1^{-1} \, \mathrm{d}(x_1^2) = 2 \int_0^\infty e^{-x_1^2} \, \mathrm{d}x_1. \tag{13}$$

Now consider the two-variable function $F(x_1, x_2) := e^{-(x_1^2 + x_2^2)}$. (A generalized) Fubini theorem applies to its integration on $A := [0, \infty) \times [0, \infty)$ and gives

$$\begin{aligned} \int_A F(x_1, x_2) \, \mathrm{d}(x_1, x_2) &= \int_0^\infty \left[\int_0^\infty e^{-x_1^2 - x_2^2} \, \mathrm{d}x_1 \right] \mathrm{d}x_2 \\ &= \left[\int_0^\infty e^{-x_1^2} \, \mathrm{d}x_1 \right] \left[\int_0^\infty e^{-x_2^2} \, \mathrm{d}x_2 \right] \\ &= \left[\int_0^\infty e^{-x^2} \, \mathrm{d}x \right]^2 = \left(\frac{\Gamma(1/2)}{2} \right)^2. \end{aligned} \tag{14}$$

Now we apply Polar coordinates to $\int_A F(x_1, x_2) \, \mathrm{d}(x_1, x_2)$:

$$\begin{aligned} \int_A F(x_1, x_2) \, \mathrm{d}(x_1, x_2) &= \int_{[0, \infty) \times [0, \pi/2]} e^{-r^2} r \, \mathrm{d}(r, \theta) \\ &= \int_0^{\pi/2} \left[\int_0^\infty e^{-r^2} r \, \mathrm{d}r \right] \mathrm{d}\theta \\ &= \frac{\pi}{4}. \end{aligned} \tag{15}$$

Note that we have applied (the generalized) Fubini again in the above calculation. Comparing (14) and (15) we see that $\Gamma(1/2) = \pm\sqrt{\pi}$. That $+$ should be chosen is obvious. □

Problem 1. Formulate and prove a generalized version of Fubini that is good enough for our need in the above example. It is OK if you only prove your "Fubini" for the particular function and the particular domains above.

Example 4. (PKU3) Let

$$A := \{(x, y, z) | \, x^2 + 2\, y^2 \leqslant z \leqslant 2 - x^2\}. \tag{16}$$

Find its volume.

First solve

$$x^2 + 2\, y^2 = 2 - x^2 \Longrightarrow x^2 + y^2 = 1 \tag{17}$$

so the volume V is given by

$$\int_B 2\,(1-x^2-y^2)\,\mathrm{d}(x,y), \qquad B:=\{(x,y)|\; x^2+y^2\leqslant 1\}. \tag{18}$$

Apply polar coordinates:

$$\begin{aligned} V &= \int_B 2\,(1-x^2-y^2)\,\mathrm{d}(x,y) \\ &= 2\int_0^{2\pi}\left[\int_0^1 (1-r^2)\,\mathrm{d}r\right]\mathrm{d}\theta \\ &= \pi. \end{aligned} \tag{19}$$

Other change of variables

Example 5. (PKU3) Calculate

$$\int_A x\,y\,\mathrm{d}(x,y) \tag{20}$$

where A is enclosed by $y^2=x, y^2=4\,x, x^2=y, x^2=4\,y$.

Set $u=y^2/x$, $v=x^2/y$. Then if T: $(u,v)\mapsto(x,y)$, we have $T^{-1}(A)=[1,4]^2$. It is easy to check that the mapping is one-to-one.

Now we have

$$\det(DT)=\frac{1}{\det(DT^{-1})}=-\frac{1}{3}, \qquad x\,y=u\,v. \tag{21}$$

Thus

$$\int_A x\,y\,\mathrm{d}(x,y)=\int_{[1,4]^2}\frac{u\,v}{3}\,\mathrm{d}(u,v)=\frac{75}{4}. \tag{22}$$

Example 6. (PKU3) Calculate

$$\int_A \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2}\mathrm{d}(x,y) \tag{23}$$

where $A:=\left\{(x,y)|\;\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1\right\}$.

We make the change of variables:

$$T(r,\theta):=\begin{pmatrix} a\,r\cos\theta \\ b\,r\sin\theta \end{pmatrix}. \tag{24}$$

Thus $T^{-1}(A)=[0,1]\times[0,2\,\pi]$. Furthermore we have

$$|\det(DT)|=a\,b\,r. \tag{25}$$

So

$$\begin{aligned} \int_A \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2}\mathrm{d}(x,y) &= \int_{[0,1]\times[0,2\pi]}\sqrt{1-r^2}\,a\,b\,r\,\mathrm{d}(r,\theta) \\ &= 2\,\pi\,a\,b\int_0^1\sqrt{1-r^2}\,r\,\mathrm{d}r \\ &= \frac{2}{3}\,\pi\,a\,b. \end{aligned} \tag{26}$$