Calculation of double integrals

Example 1. Let $I := [1,2] \times [0,1]$. Calculate

$$\int_I \frac{x^2}{1+y^2}\,\mathrm{d}(x,y). \tag{1}$$

Since $f(x,y) := \frac{x^2}{1+y^2}$ is continuous, we can apply Fubini:

$$\begin{aligned}\int_I f(x,y)\mathrm{d}(x,y) &= \int_1^2 \left[\int_0^1 \frac{x^2}{1+y^2}\,\mathrm{d}y\right]\mathrm{d}x \\ &= \int_1^2 x^2\,(\arctan y)|_0^1\,\mathrm{d}x \\ &= \int_1^2 x^2\left(\frac{\pi}{4}-0\right)\mathrm{d}x \\ &= \frac{\pi}{4}\int_1^2 x^2\,\mathrm{d}x \\ &= \frac{7\,\pi}{12}.\end{aligned} \tag{2}$$

Example 2. Consider E be encircled by $y=0, x=1$ and $y=x$. Calculate

$$\int_A \sqrt{4\,x^2-y^2}\,\mathrm{d}(x,y). \tag{3}$$

Define

$$f(x,y) = \begin{cases} \sqrt{4\,x^2-y^2} & (x,y)\in A \\ 0 & (x,y)\notin A \end{cases}. \tag{4}$$

Take $I{:=}[0,1]\times[0,1]$. Then $f(x,y)$ satisfies the conditions of Fubini and we have

$$\begin{aligned}\int_A \sqrt{4\,x^2-y^2}\,\mathrm{d}(x,y) &= \int_I f(x,y)\,\mathrm{d}(x,y) \\ &= \int_0^1\left[\int_0^1 f(x,y)\,\mathrm{d}y\right]\mathrm{d}x \\ &= \int_0^1\left[\int_0^x \sqrt{4\,x^2-y^2}\,\mathrm{d}y\right]\mathrm{d}x \\ &= \int_0^1\left[2\,x\int_0^x \sqrt{1-\left(\frac{y}{2\,x}\right)^2}\,\mathrm{d}y\right]\mathrm{d}x \\ &= \int_0^1\left[4x^2\int_0^{1/2}\sqrt{1-z^2}\,\mathrm{d}z\right]\mathrm{d}x \\ &= \int_0^1\left[4\,x^2\int_0^{\pi/6}\cos\theta\,\mathrm{d}\sin\theta\right]\mathrm{d}x \\ &= \int_0^1\left[4\,x^2\int_0^{\pi/6}(\cos\theta)^2\,\mathrm{d}\theta\right]\mathrm{d}x \\ &= \int_0^1\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)x^2\,\mathrm{d}x = \frac{\pi}{9}+\frac{\sqrt{3}}{6}.\end{aligned} \tag{5}$$

Exercise 1. Explain the red parts of the above example.

Remark 3. If we integrate x first, we would have

$$\int_0^1 \left[\int_y^1 \sqrt{4\,x^2 - y^2}\, \mathrm{d}x \right] \mathrm{d}y = \frac{1}{2} \int_0^1 \left[\frac{1}{2} u \sqrt{u^2 - y^2} - \frac{y^2}{2} \ln \left(u + \sqrt{u^2 - y^2} \right) \right]_{u=2y}^{u=2} \mathrm{d}y \tag{6}$$

We see that choosing the correct order of integration is important.

Example 4. Let A be encircled by $y = x, y = x + 2, y = 2, y = 6$. Calculate

$$\int_A (x^2 + y^2)\, \mathrm{d}(x, y). \tag{7}$$

We have

$$\begin{aligned} \int_A (x^2 + y^2)\, \mathrm{d}(x, y) &= \int_I f(x, y)\, \mathrm{d}(x, y) \\ &= \int_2^6 \left[\int_{y-2}^{y} (x^2 + y^2)\, \mathrm{d}x \right] \mathrm{d}y \\ &= 224. \end{aligned} \tag{8}$$

Example 5. Let A be the region between $x^2 + y^2 = 4$ and $y = -x^2 + 1, y = x^2 - 1$. Calculate

$$\int_A (x^2 + y)\, \mathrm{d}(x, y). \tag{9}$$

We can calculate that the integral is $4\,\pi - \frac{8}{15}$ through dividing:

$$A = \cup_{i=1}^4 A_i \tag{10}$$

where

$$A_1 := \left\{ (x, y) |\, -2 \leqslant x \leqslant -1, -\sqrt{4 - x^2} \leqslant y \leqslant \sqrt{4 - x^2} \right\}; \tag{11}$$
$$A_2 := \left\{ (x, y) |\, 1 \leqslant x \leqslant 2, -\sqrt{4 - x^2} \leqslant y \leqslant \sqrt{4 - x^2} \right\}; \tag{12}$$
$$A_3 := \left\{ (x, y) |\, -1 \leqslant x \leqslant 1, -\sqrt{4 - x^2} \leqslant y \leqslant x^2 - 1 \right\}; \tag{13}$$
$$A_4 := \left\{ (x, y) |\, -1 \leqslant x \leqslant 1, -x^2 + 1 \leqslant y \leqslant \sqrt{4 - x^2} \right\}. \tag{14}$$

Exercise 2. Let A be a triangle in $\mathbb{R}^2$ with vertices $(a_1, b_1), (a_2, b_2), (a_3, b_3)$. Prove that its area is

$$\frac{1}{2} \det \begin{pmatrix} 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \\ 1 & a_3 & b_3 \end{pmatrix}. \tag{15}$$