Zero measure set

If A is a simple graph, then any reasonable measure theory would give us the same value for $\mu(A)$. However this is not the case anymore once we try to assign measures for sets more comlicated than simple graphs. Therefore from now on we will call our measure “Jordan measure” (also called “content” in some books).

One class of sets for which their Jordan measures can be readily constructed is those with measure 0.

Definition 1. (Zero measure set) *A set $A \subseteq \mathbb{R}^N$ has Jordan measure zero if and only if for any $\varepsilon > 0$ there is a simple graph B such that*

$$\bar{A} \subseteq B \text{ and } \mu(B) < \varepsilon. \tag{1}$$

Exercise 1. Let $A \subseteq B \subseteq \mathbb{R}^N$. Then $\mu(A) = 0 \Longrightarrow \mu(B) = 0$.

The following theorem makes checking zero measure-ness much easier.

Theorem 2. *A set $A \subseteq \mathbb{R}^N$ has Jordan measure zero if and only if $\bar{A}$ has Jordan measure zero. That is for each $\varepsilon > 0$, there is a simple graph B such that*

$$\bar{A} \subseteq B \text{ and } \mu(B) < \varepsilon. \tag{2}$$

Proof.

- If. The condition in the theorem shows $\mu(\bar{A}) = 0$. Since $A \subseteq \bar{A}$, $\mu(A) = 0$.
- Only if. Let $A \subseteq \mathbb{R}^N$ be such that $\mu(A) = 0$. That is for each $\varepsilon > 0$, there is a simple graph B such that

  $$\bar{A} \subseteq B \text{ and } \mu(B) < \varepsilon. \tag{3}$$

  By definition B is closed. Therefore $\bar{\bar{A}} \subseteq B$ and the conclusion follows. □

Exercise 2. Is it true that $\mu(A^o) = 0 \Longrightarrow \mu(A) = 0$?

Example 3. Let $E \subseteq \mathbb{R}^N$ be a finite set. Then E is a measure zero set.

Proof. Let $E = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ be a finite set. Define intervals $I_k = \{\boldsymbol{x}_k\}$. □

Example 4. Consider $\mathbb{N} \subseteq \mathbb{R}$. Is $\mathbb{N}$ a measure zero set?

Solution. Let $I_1, \ldots, I_n$ be intervals such that

$$\mathbb{N} \subseteq I_1 \cup \cdots \cup I_n. \tag{4}$$

Then at least one of I_n must be unbounded and therefore $\mathbb{N}$ is not a measure zero set.

Example 5. Let $A := \{1/n |\, n \in \mathbb{N}\} \subset \mathbb{R}$. Is A a measure zero set?

Solution. For any $\varepsilon > 0$, take $0 < \delta < \varepsilon$. There is $N \in \mathbb{N}$ such that $N > \delta$[1]. Then we have

$$\forall n \geqslant N, \qquad \frac{1}{n} \in [0, \delta]. \tag{5}$$

Set $I_N := [0, \delta/2]$.[2] Now define for $k = 1, 2, \ldots, N - 1$,

$$I_k := \left[\frac{1}{k}, \frac{1}{k}\right]. \tag{6}$$

Then

$$A \subseteq \cup_{k=1}^{N} I_k \tag{7}$$

and

$$\sum_{k=1}^{N} \mu(I_k) = \mu(I_N) = \delta < \varepsilon \tag{8}$$

Exercise 3. Let $A := \left\{\left(\frac{1}{m}, \frac{1}{n}\right) | \, m, n \in \mathbb{N}\right\}$. Prove that A is a measure zero set.

Example 6. Let $A := [0, 1] \cap \mathbb{Q}$. Is A a measure zero set?

Solution. Since $\bar{A} = [0, 1]$ is a simple graph, we have $\mu(\bar{A}) = 1 \neq 0$. Therefore A does not have measure zero.

Exercise 4. Prove that the unit circle in $\mathbb{R}^2$ has Jordan measure 0.

Example 7. Let $f: \mathbb{R} \mapsto \mathbb{R}$ be continuous. Let $A := \{(x, f(x)) | \, x \in [a, b]\}$. Then $\mu(A) = 0$.

Soluton. Since $[a, b]$ is compact, f is uniformly continuous.

For any $\varepsilon > 0$, take $\delta > 0$ such that

$$\forall |x - y| < \delta, \qquad |f(x) - f(y)| < \frac{\varepsilon}{b - a}. \tag{9}$$

Now take $N \in \mathbb{N}$ such that $N\delta > b - a$. Let $h := \frac{b-a}{N}$ and define

$$x_0 = a, \;\; x_1 = a + h, \ldots, x_{N-1} = b - h, \;\; x_N = b. \tag{10}$$

Then since $h < \delta$ we have

$$\max_{x \in [x_i, x_{i+1}]} f - \min_{x \in [x_i, x_{i+1}]} f < \frac{\varepsilon}{b - a}. \tag{11}$$

Now define

$$I_k := [x_{k-1}, x_k] \times \left[\min_{x \in [x_{k-1}, x_k]} f, \max_{x \in [x_{k-1}, x_k]} f\right]. \tag{12}$$

We have

$$A \subseteq \cup_{k=1}^{N} I_k \tag{13}$$

1. $N > \delta^{-1}$.
2. $[0, \delta]$.

and furthermore

$$\mu(I_k) < \frac{h\,\varepsilon}{b-a} \Longrightarrow \sum_{k=1}^{N} \mu(I_k) < \frac{N\,h\,\varepsilon}{b-a} = \varepsilon. \tag{14}$$

Exercise 5. Let $f: \mathbb{R} \mapsto \mathbb{R}$ be Riemann integrable on $[a, b]$. Let $A := \{(x, f(x)) | x \in [a, b]\}$. Then $\mu(A) = 0$.

Lemma 8. *Let* $f: \mathbb{R}^N \mapsto \mathbb{R}$ *be differentiable and* $A := \{\boldsymbol{x} \in \mathbb{R}^N | f(\boldsymbol{x}) = 0\}$. *Further assume that* $(\text{grad}\, f)(\boldsymbol{x}) \neq 0$ *for every* $\boldsymbol{x} \in A$. *Then* $\mu(A) = 0$.

Exercise 6. Prove the above lemma.

Exercise 7. Let $A_1, \ldots, A_n$ be such that $\mu(A_k) = 0$, $k = 1, 2, 3, \ldots, n$. Let $A = \cup_{k=1}^n A_k$. Then $\mu(A) = 0$.

Exercise 8. Let $\{A_n\}_{n=1}^\infty$ be such that $\mu(A_n) = 0$ for all n. Let $A = \cup_{n=1}^\infty A_n$. Do we have $\mu(A) = 0$? Justify your answer.

Lemma 9. *Let* I *be a bounded closed interval in* $\mathbb{R}^N$. *Then* $\mu(\partial I) = 0$.

Proof. Just notice that ∂I is the union of finitely many pieces of graphs of constant functions. □

Lemma 10. *Let* $A \subseteq \mathbb{R}^N$. *Then* $\mu(A) \neq 0$ *if and only if there is* $\varepsilon_0 > 0$ *such that for all collections of bounded closed intervals* $\{I_k\}_{k=1}^n$ *covering* A, *we have*

$$\sum_{k=1, I_k^o \cap A \neq \varnothing}^{n} \mu(I_k) \geqslant \varepsilon_0. \tag{15}$$

Remark 11. Note that the summation is only over those I_k's such that $I_k^o \cap A \neq \varnothing$.

Proof. We prove by contradiction. Assume that for any $\varepsilon > 0$, there is $\{I_k\}_{k=1}^n$ $(n = n(\varepsilon))$ such that

$$\sum_{k=1, I_k^o \cap A \neq \varnothing}^{n} \mu(I_k) < \varepsilon. \tag{16}$$

Denote $\delta = \varepsilon - \sum_{k=1, I_k^o \cap A \neq \varnothing}^n \mu(I_k)$.

Re-labeling the intervals, we assume $I_k^o \cap A = \varnothing$ for $k = 1, 2, \ldots, m$. Since $\partial I_k \cup I_k^o = I_k$, we have

$$A \subseteq (\cup_{k=m+1}^n I_k) \cup (\cup_{k=1}^m \partial I_k). \tag{17}$$

But $\cup_{k=1}^m \partial I_k$ has measure zero, therefore there are compact intervals $J_1, \ldots, J_p$ such that

$$(\cup_{k=1}^m \partial I_k) \subseteq \cup_{i=1}^p J_p \tag{18}$$

and

$$\sum_{i=1}^{p} \mu(J_p) < \delta. \tag{19}$$

Now we have

$$A \subseteq (\cup_{k=m+1}^n I_k) \cup (\cup_{i=1}^p J_p) \tag{20}$$

with

$$\sum_{k=m+1}^{n} \mu(I_k) + \sum_{i=1}^{p} \mu(J_p) = \sum_{k=1, I_k^o \cap A \neq \varnothing}^{n} \mu(I_k) + \sum_{i=1}^{p} \mu(J_p) < \varepsilon. \tag{21}$$

Thus $\mu(A) = 0$. Contradiction. □

Problem 1. Let $\boldsymbol{l}: \mathbb{R}^N \mapsto \mathbb{R}^N$ be linear with matrix representation $A \in \mathbb{R}^{N \times N}$. Then

$$\det A = 0 \Longleftrightarrow \forall E \in \mathbb{R}^N, \qquad \mu(\boldsymbol{l}(E)) = 0. \tag{22}$$

Problem 2. Let $A \subseteq \mathbb{R}^N$ and $\boldsymbol{f}: \mathbb{R}^N \mapsto \mathbb{R}^N$. Assume $\mu(A) = 0$ and $\boldsymbol{f}$ is continuous. Then $\mu(\boldsymbol{f}(A)) = 0$. (Note that in particular this means $\mu(O\ A) = 0$ for any orthogonal transformation O). (Is the claim false? Get a counter-example? – Peano's curve? Or any space-filling curve...)