Measuring simple graphs

Simple graphs

We introduce that idea of simple graphs:

Definition 1. *$A \subseteq \mathbb{R}^N$ is a "simple graph" if there are compact intervals $I_1, \ldots, I_n$ such that $A = \cup_{k=1}^n I_k$.*

Theorem 2. *Let A, B be simple graphs. Let $\boldsymbol{x}_0 \in \mathbb{R}^N$, $a \in \mathbb{R}$. Then so are $a\,A$, $A + \boldsymbol{x}_0$, $A \cap B, A \cup B, A - B^o$.*

Proof. As A, B are simple graphs, there are compact intervals $I_1, \ldots, I_n$ and $J_1, \ldots, J_m$ such that

$$A = \cup_{k=1}^n I_k, \qquad B = \cup_{l=1}^m J_l. \tag{1}$$

- $a\,A$. We have

  $$a\,A = \cup_{k=1}^n (a\,I_k) \tag{2}$$

  and the conclusion follows from the fact that $a\,I_k$ is also a compact interval.

- $A + \boldsymbol{x}_0$. We have

  $$A + \boldsymbol{x}_0 = \cup_{k=1}^n (I_k + \boldsymbol{x}_0) \tag{3}$$

  and the conclusion follows from the fact that $I_k + \boldsymbol{x}_0$ is also a compact interval.

- $A \cap B$. We have

  $$A \cap B = [\cup_{k=1}^n I_k] \cap [\cup_{l=1}^m J_l] = \cup_{k=1}^n \cup_{l=1}^m (I_k \cap J_l) \tag{4}$$

  Thus all we need to show is that $I_k \cap J_l$ is a compact interval.

  Let $I_k = \{\boldsymbol{x} \in \mathbb{R}^N |\ x_1 \in [a_1, b_1], x_2 \in [a_2, b_2], \ldots, x_N \in [a_N, b_N]\}$ and $J_l = \{\boldsymbol{x} \in \mathbb{R}^N |\ x_1 \in [c_1, d_1], \ldots, x_N \in [c_N, d_N]\}$. Then we clearly have

  $$I_k \cap J_l = \{\boldsymbol{x} \in \mathbb{R}^N |\ x_1 \in [a_1, b_1] \cap [c_1, d_1], \ldots, x_N \in [a_N, b_N] \cap [c_N, d_N]\} \tag{5}$$

  which is still a compact interval.

- $A \cup B$. We have

  $$A \cup B = [\cup_{k=1}^n I_k] \cup [\cup_{l=1}^m J_l] \tag{6}$$

  so it is a simple graph.

- $A - B^o$. Note that by our definition $(B^o)^c$ is not a simple graph so we cannot use $A - B^o = A \cap (B^o)^c$. Instead, as A, B are bounded, there is a compact interval I such that $A \cup B \subseteq I$. Now all we need to prove is that $I - B^o$ is a simple graph. Without loss of generality, we assume $I = [-R, R]^N$ for some $R > 0$.

  Now assume $B = \cup_{k=1}^n I_n$. Note that $B^o \neq \cup_{k=1}^n I_n^o$. Now consider the end points in x_1 direction for each I_n, we have $2\,n$ such points. Denote them by $x_1^1 \leqslant x_2^1 \leqslant \ldots \leqslant x_{2n}^1$; Similarly we obtain $x_1^i, \ldots, x_{2n}^i$ for every $i = 2, 3, \ldots, N$. Now let $x_0^i = -R, x_{2n+1}^i = R$. If we consider the following intervals

  $$J_{i_1 i_2 \ldots i_N} := [x_{i_1}, x_{i_1+1}] \times \cdots \times [x_{i_N}, x_{i_N+1}] \tag{7}$$

for some $i_1, \ldots, i_N \in \{0, 2, \ldots, 2n\}$, then we see that

$$I - B = \cup J_{i_1 \ldots i_N} \tag{8}$$

where the union is taken over all $J_{i_1 \cdots i_N}$ satisfying $J^o_{i_1 \cdots i_N} \cap B = \varnothing$. Therefore $I - B$ is a simple graph. □

Remark 3. Note that in general rotating a simple graph does not give a simple graph anymore.

Exercise 1. Prove that if $A \cup B \subseteq I$ then $A - B = A \cap (I - B)$.

Exercise 2. Explain why in general $B^o \neq \cup_{k=1}^n I_n^o$.

Corollary 4. *Let $A_1, \ldots, A_n$ be simple graphs. Then so are $\cap_{k=1}^n A_k$ and $\cup_{k=1}^n A_k$.*

Theorem 5. *Let A be a simple graph. Then there are compact intervals $I_1, \ldots, I_n$ satisfying $I_i^o \cap I_j^o = \varnothing$ for any $i \neq j$, such that*

$$A = \cup_{k=1}^n I_k. \tag{9}$$

Proof. The idea is similar to that in the proof for $A - B^o$ in the last theorem. □

Measuring simple graphs

We will try to define a measure for all simple graphs. Thanks to Theorem 5 it suffices to define $\mu(I)$ for every compact interval I.

Definition 6. (Measure for compact intervals) *Let $I = [a_1, b_1] \times \cdots \times [a_N, b_N]$ be a compact interval in $\mathbb{R}^N$. We define it's Jordan measure to be*

$$\mu(I) := (b_1 - a_1) \cdots (b_N - a_N). \tag{10}$$

Remark 7. This definition is more subtle than it looks. Explore to see whether this follows from i–v.

Definition 8. (Measure for simple graphs) *Let A be a simple graph. Let $I_1, \ldots, I_n$ be compact intervals satisfying $I_i^o \cap I_j^o = \varnothing$ for any $i \neq j$ and $A = \cup_{k=1}^n I_k$. Define*

$$\mu(A) = \sum_{k=1}^n \mu(I_k). \tag{11}$$

Theorem 9. (Consistency) *The above definition is consistent, that is $\mu(A)$ is independent of the choice of $I_1, \ldots, I_n$. In other words, if $I_1, \ldots, I_n$ and $J_1, \ldots, J_m$ satisfy $I_i^o \cap I_j^o = \varnothing$, $J_p^o \cap J_q^o = \varnothing$, and furthermore $A = \cup_{k=1}^n I_k = \cup_{l=1}^m J_l$, then*

$$\sum_{k=1}^n \mu(I_k) = \sum_{l=1}^m \mu(J_l). \tag{12}$$

Proof. Omitted. □

Remark 10. Furthermore, note that here we are claiming that all simple graphs are measurable. To back up this claim, we need to check the consistency of the definition with i – v.

Theorem 11. *Let μ be defined for all simple graphs as in Definition 8. Then within the set of all simple graphs, it satisfies i – v.*

Proof.

i. Lineariy. Let A, B be simple graphs. Then there are compact intervals $I_1, \ldots, I_n$, $J_1, \ldots, J_m$ such that $I_i^o \cap I_j^o = \varnothing$, $J_p^o \cap J_q^o = \varnothing$, and furthermore $A = \cup_{k=1}^n I_k$, $B = \cup_{l=1}^m J_l$. Now we have

$$A \cup B = (\cup_{k=1}^n I_k) \cup (\cup_{l=1}^m J_l). \tag{13}$$

Thus all we need to show is $I_k^o \cap J_l^o = \varnothing$ which follows from $A \cap B = \varnothing$.

ii. Monotonicity: Let $A \subseteq B$. Let $C := B - A$. It is also a simple graph and satisfies $C \cap A = \varnothing$. We have

$$\mu(B) = \mu(A) + \mu(C) \geqslant \mu(A). \tag{14}$$

iii. Translation and rotation invariance. Note that rotation invariance doesn't apply here. The translation part is left as exercise.

iv. Homogeneity. Exercise.

v. Normalized. Trivial. □

The following is a different way of defining the measure for simple graphs (in some sense this is "stability" of the measure).

Theorem 12. *Let A be a simple graph. Let $W_{\text{out}} := \{B \subseteq \mathbb{R}^N | \ B$ is a simple graph and $A \subseteq B\}$, $W_{\text{in}} := \{C \subseteq \mathbb{R}^N | \ C$ is a simple graph and $C \subseteq A\}$. Then*

$$\mu(A) = \inf_{B \in W_{\text{out}}} \mu(B) = \sup_{C \in W_{\text{in}}} \mu(C). \tag{15}$$

Proof. By monotonicity of μ, we have

$$\forall B \in W_{\text{out}}, \qquad \mu(A) \leqslant \mu(B). \tag{16}$$

On the other hand, $A \in W_{\text{out}}$. Therefore $\mu(A) = \inf_{B \in W_{\text{out}}} \mu(B)$. Similarly we can prove the other equality. □

Theorem 13. *Let A be a simple graph, define $W_{\text{in}}' := \{C \subseteq \mathbb{R}^N | \ C$ is a simple graph and $C \subseteq A^o\}$. Then*

$$\mu(A) = \sup_{C \in W_{\text{in}}'} \mu(C). \tag{17}$$

Proof. Since A is a simple graph, there are compact intervals I_1, ..., I_n such that $I_i^o \cap I_j^o = \varnothing$ and $\mu(A) = \sum_{k=1}^n \mu(I_k)$. Take any $\boldsymbol{x}_k \in I_k$ and any $a \in (0, 1)$. Define

$$J_k := a\,(I_k - \boldsymbol{x}_k) + \boldsymbol{x}_k. \tag{18}$$

Then $J_k \subseteq I_k^o$ which implies

$$C := \cup_{k=1}^n J_k \subseteq A^o. \tag{19}$$

As $I_i^o \cap I_j^o = \varnothing$, we have $J_i^o \cap J_j^o = \varnothing$ and consequently

$$\mu(C) = \sum_{k=1}^n \mu(J_k) = a^N \mu(A). \tag{20}$$

Clearly $C \in W_{\text{in}}$ so

$$\sup_{B \in W_{\text{in}}'} \mu(B) \geqslant \mu(C) = a^N \mu(A). \tag{21}$$

By arbitrariness of a we have

$$\sup_{B \in W_{\text{in}}'} \mu(B) \geqslant \mu(A) \tag{22}$$

On the other hand $\sup_{B \in W_{\text{in}}} \mu(B) \leqslant \mu(A)$ so the conclusion follows. □

Exercise 3. Prove $J_k \subseteq I_k^o$ and then $C \subseteq A^o$.

Exercise 4. Prove that if for all $a \in (0, 1)$ we have $\sup_{B \in W_{\text{in}}} \mu(B) \geqslant a^N \mu(A)$, then necessarily $\sup_{B \in W_{\text{in}}} \mu(B) \geqslant \mu(A)$.