Heine-Borel Theorem

Heine-Borel Theorem completely characterizes compact sets in $\mathbb{R}^N$.

Lemma 1. *If $E \subseteq \mathbb{R}^N$ is compact, then it is closed.*

Proof. We try to prove E^c is open. Take any $\boldsymbol{x} \notin E$. Consider the family of open sets

$$\{(\overline{B(\boldsymbol{x}, 1/n)})^c | \, n \in \mathbb{N}\}. \tag{1}$$

whose union is $\mathbb{R}^n - \{\boldsymbol{x}\} \supseteq E$. Thus there is a finite cover. □

Exercise 1. Complete the above proof.

Lemma 2. *If $E \subseteq \mathbb{R}^N$ is compact, then it is bounded.*

Proof. Left as exercise. □

Exercise 2. Prove the above lemma.

Lemma 3. *If $E \subseteq \mathbb{R}^N$ is bounded and closed, then E is compact.*

Proof. In light of Lemma ?, it suffices to show the compactness of closed intervals. In the following we take $E = I$ to be a closed interval.

Let W be an open covering of I that does not have a finite subcover. Bi-secting I into 2^N closed subintervals as in the proof of Theorem ?, at least one of them, denote it by I_1, cannot be covered by finitely many open sets from W. Do this again and again, we have a sequence of nested closed intervals $I \supseteq I_1 \supseteq I_2 \supseteq \cdots$, none of the I_k's can be covered by finitely many open sets from W. By Nested Intervals Theorem we know there is $\boldsymbol{x}_0 \in I$ such that $\{\boldsymbol{x}_0\} = \cap_{n=1}^{\infty} I_n$.

Now because W is an open covering of I, there is an open set $U \in W$ such that $\boldsymbol{x}_0 \in U$. Consequently there is $r > 0$ such that $B(\boldsymbol{x}_0, r) \subseteq U$. Now by construction there is $n \in \mathbb{N}$ such that $I_n \subset B(\boldsymbol{x}_0, r) \subseteq U$ which means I_n can be covered by a single open set from W. Contradiction. □

Exercise 3. Consider the set of all sequences of real numbers, turned into a inner product space through natural addition/subtraction/scalar multiplication/inner product:

$$\{x_n\} \pm \{y_n\} := \{x_n \pm y_n\}; \qquad a\,\{x_n\} := \{a\,x_n\} \tag{2}$$

$$\{x_n\} \cdot \{y_n\} := \sum_{n=1}^{\infty} x_n\,y_n = x_1\,y_1 + x_2\,y_2 + \cdots. \tag{3}$$

Then we can define the l^2 norm as:

$$\|\{x_n\}\| := (\{x_n\} \cdot \{x_n\})^{1/2}. \tag{4}$$

The definition of open balls and open sets now can be carried out exactly the same as in $\mathbb{R}^N$. This is the intuitive genearlization of $\mathbb{R}^N$ to the case $N = \infty$. This infinite dimensional Euclidean space is denoted l^2.

a) Give definition of open, closed, compact, bounded sets in l^2.

b) Prove that if $A \subseteq l^2$ is compact, then A is bounded and closed.

c) Find a bounded and closed set that is not compact. Justify your answer.

Summarizing the above three lemmas, we reach the following theorem.

Theorem 4. (Heine-Borel) *Let $K \subseteq \mathbb{R}^N$. Then*

$$K \text{ is compact} \Longleftrightarrow K \text{ is closed and bounded.} \tag{5}$$

Compactness and convergent subsequences

Theorem 5. *Let $K \subseteq \mathbb{R}^N$. Then K is compact $\Longleftrightarrow$ Every sequence in K has a convergent subsequence whose limit is in K.*

Proof.

$\Longrightarrow$. By Heine-Borel, K is compact $\Longrightarrow$ K is bounded and closed. Now if the sequence as a set in $\mathbb{R}^N$ only has finitely many points, the claim is obvious; Otherwise we apply Bolzano-Weierstrass to the set $\{\boldsymbol{x}_n\}$ to obtain the existence of at least one cluster point $\boldsymbol{x}_0$. Then there is a subsequence converging to $\boldsymbol{x}_0$ (see the exercise below). Finally, since K is closed, $\boldsymbol{x}_0 \in K$.

$\Longleftarrow$. Thanks to Heine-Borel we only need to show K is closed and bounded. Both are easy and left as exercises. □

Exercise 4. Let $\{\boldsymbol{x}_n\}$ be a sequence in $\mathbb{R}^N$. Denote by A the set $\{\boldsymbol{x}_n\}$. Prove that if $\boldsymbol{x}_0$ is a cluster point of A, then there is a subsequence $\boldsymbol{x}_{n_k} \longrightarrow \boldsymbol{x}_0$.

Exercise 5. Let $K \subseteq \mathbb{R}^N$. Prove that if every sequence in K has a convergent subsequence, then K must be bounded.

Exercise 6. Let $K \subseteq \mathbb{R}^N$. Prove that if every sequence in K has a convergent subsequence whose limit is in K, then K must be closed.

Exercise 7. Let $K \subseteq \mathbb{R}^N$. Then K is compact if and only if any infinite subset $A \subseteq K$ has a cluster point $\boldsymbol{x} \in K$.

Further study of compactness

Properties of compact sets

The following is a generalization of Nested Intervals Theorem.

Theorem 6. (Nested Compact Sets) *Let $\{K_n\}$ be a sequence of nonempty compact sets in $\mathbb{R}^N$ satisfying $K_1 \supseteq K_2 \supseteq K_3 \cdots$. Then $\cap_{n=1}^{\infty} K_n \neq \varnothing$.*

Proof. Assume otherwise. Then $\{K_n^c\}$ is an open cover of K_1. Since K_1 is compact, there is a finite subcover: $K_1 \subseteq K_{n_1}^c \cup K_{n_2}^c \cup \cdots \cup K_{n_m}^c$. Now note that since $K_1 \supseteq K_2 \supseteq K_3 \cdots$, $K_1^c \subseteq K_2^c \subseteq \cdots$ which means

$$K_1 \subseteq K_{n_1}^c \cup K_{n_2}^c \cup \cdots \cup K_{n_m}^c = K_{n_m}^c \tag{6}$$

This leads to $K_1 \cap K_{n_m} = \varnothing$. Together with $K_{n_m} \subseteq K_1$ we conclude $K_{n_m} = \varnothing$. Contradiction. □

Exercise 8. From the proof it seems that the theorem can be generalized as follows:

Let $\{K_n\}$ be a sequence of closed sets in $\mathbb{R}^N$ satisfying $K_1 \supseteq K_2 \supseteq K_3 \cdots$ with K_1 compact. Then $\cap_{n=1}^{\infty} K_n \neq \varnothing$.

Is this really a generalization? Justify your answer.

Theorem 7. *Let E, F be compact sets. Then* $\operatorname{dist}(E, F) > 0 \Longleftrightarrow E \cap F = \varnothing$.

Proof.

$\Longrightarrow$. This is trivial.

$\Longleftarrow$. As $E \cap F = \varnothing$, $E \subseteq F^c$. Note that F^c is open. Now for any $\boldsymbol{x} \in E$, there is $r = r(\boldsymbol{x}) > 0$ such that $B(\boldsymbol{x}, r) \subseteq F^c$. We have an oper covering of E:

$$E \subset \cup_{\boldsymbol{x} \in E} B\left(\boldsymbol{x}, \frac{r(\boldsymbol{x})}{2}\right). \tag{7}$$

By compactness of E there is a finite subcovering:

$$E \subset B\left(\boldsymbol{x}_1, \frac{r_1}{2}\right) \cup \cdots \cup B\left(\boldsymbol{x}_n, \frac{r_n}{2}\right). \tag{8}$$

Take $d := \min \left\{\frac{r_1}{2}, \ldots, \frac{r_n}{2}\right\}$. Now for any $\boldsymbol{x} \in E$, there is $k \in \{1, 2, \ldots, n\}$ such that $\boldsymbol{x} \in B\left(\boldsymbol{x}_k, \frac{r_k}{2}\right)$. Now $B(\boldsymbol{x}_k, r_k) \subseteq F^c \Longrightarrow F \subseteq B(\boldsymbol{x}_k, r_k)^c = \{\boldsymbol{y} \in \mathbb{R}^N | \|\boldsymbol{y} - \boldsymbol{x}_k\| \geqslant r_k\}$. As a consequence we have $\|\boldsymbol{x} - \boldsymbol{y}\| \geqslant \frac{r_k}{2} \geqslant d$ (see the following exercise). Note that this holds for any $\boldsymbol{x} \in E, \boldsymbol{y} \in F$. Therefore $\operatorname{dist}(E, F) \geqslant d > 0$ and the proof ends. □

Exercise 9. Let $\boldsymbol{x} \in B(\boldsymbol{x}_0, r_1)$ and $\boldsymbol{y} \in B(\boldsymbol{x}_0, r_2)^c$. Prove that $\|\boldsymbol{x} - \boldsymbol{y}\| \geqslant \max\{0, r_2 - r_1\}$.

Exercise 10. Let $E, F \subseteq \mathbb{R}^N$ be closed. Further assume E is compact. Prove that $\operatorname{dist}(E, F) > 0 \Longleftrightarrow E \cap F = \varnothing$.

Exercise 11. Find two closed sets $E, F \in \mathbb{R}^N$ such that $E \cap F = \varnothing$ but $\operatorname{dist}(E, F) = 0$.

Exercise 12. Let E be compact and F be closed. Prove that there are $\boldsymbol{x}_0 \in E$, $\boldsymbol{y}_0 \in F$ such that $\operatorname{dist}(\boldsymbol{x}_0, \boldsymbol{y}_0) := \|\boldsymbol{x}_0 - \boldsymbol{y}_0\| = \operatorname{dist}(E, F)$. Does the claim still hold if both E, F are only closed? Justify your answer.

Continuous functions on compact sets

Theorem 8. *Let $\boldsymbol{f}: \mathbb{R}^N \mapsto \mathbb{R}^M$ be continuous. Let $E \subseteq \mathbb{R}^N$ be compact. Then $\boldsymbol{f}(E)$ is compact.*

Proof. Take any open covering W of $\boldsymbol{f}(E)$. As $\boldsymbol{f}$ is continuous, $\boldsymbol{f}^{-1}(U)$ is open for any $U \in W$. So $\{\boldsymbol{f}^{-1}(U) | U \in W\}$ is an open covering of E. But E is compact so there is a finite subcovering:

$$E \subseteq \boldsymbol{f}^{-1}(U_1) \cup \cdots \cup \boldsymbol{f}^{-1}(U_n). \tag{9}$$

As $\boldsymbol{f}(\boldsymbol{f}^{-1}(U_k)) \subseteq U_k$, we see that $\boldsymbol{f}(E) \subseteq U_1 \cup \cdots \cup U_n$. □

Exercise 13. Find a discontinuous function f such that E compact $\Longrightarrow f(E)$ compact.

Exercise 14. Let $f: \mathbb{R}^N \mapsto \mathbb{R}$ be continuous, and $E \subset \mathbb{R}^N$ compact. Then there are $\boldsymbol{x}_{\max}, \boldsymbol{x}_{\min} \in E$ such that

$$\forall \boldsymbol{x} \in E, \qquad f(\boldsymbol{x}_{\max}) \geqslant f(\boldsymbol{x}) \geqslant f(\boldsymbol{x}_{\min}). \tag{10}$$

That is f reaches its maximum and minimum. Try to prove this in one line using the above theorem.

Give a counterexample for E bounded but not compact.

Exercise 15. Let $A := \{(x, y) | |x| + |y| \leqslant 1\}$ and $B := \{(x, y) | |x| + |y| < 1\}$. Prove that there can be no function $\boldsymbol{f}: A \mapsto B$ that is both continuous and onto.

Uniform continuity

Theorem 9. (Uniform continuity) *Let* $\boldsymbol{f}\colon \mathbb{R}^N \mapsto \mathbb{R}^M$ *be continuous. Let* $E \subseteq \mathbb{R}^N$ *be compact. Then* $\boldsymbol{f}$ *is uniformly continuous on* E*, that is for any* $\varepsilon > 0$*, there is* $\delta > 0$ *such that*

$$\forall \boldsymbol{x}, \boldsymbol{y} \in E, \ \|\boldsymbol{x} - \boldsymbol{y}\| < \delta \Longrightarrow \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| < \varepsilon. \tag{11}$$

Proof. Let $\varepsilon > 0$ be arbitrary. Then at each $\boldsymbol{x} \in E$, there is $\delta = \delta(\boldsymbol{x}) > 0$ such that

$$\forall \boldsymbol{y} \in E, \ \|\boldsymbol{x} - \boldsymbol{y}\| < \delta(\boldsymbol{x}) \Longrightarrow \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| < \varepsilon. \tag{12}$$

The balls $B\left(\boldsymbol{x}, \frac{\delta(\boldsymbol{x})}{2}\right)$ then form an open covering of E. Since E is compact, there is a finite subcovering

$$E \subseteq B\left(\boldsymbol{x}_1, \frac{\delta_1}{2}\right) \cup \cdots \cup B\left(\boldsymbol{x}_n, \frac{\delta_n}{2}\right). \tag{13}$$

Take $\delta := \frac{1}{2} \min \{\delta_1, \ldots, \delta_n\}$. Then for any $\boldsymbol{x}, \boldsymbol{y}$ such that $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta$, there must be one $k \in \{1, \ldots, n\}$ such that $\boldsymbol{x}, \boldsymbol{y} \in B(\boldsymbol{x}_k, \delta_k)$ (exercise). Consequently $\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| < \varepsilon$ and proof ends. $\square$

Exercise 16. Prove the claim "Then for any $\boldsymbol{x}$, $\boldsymbol{y}$ such that $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta$, there must be one $k \in \{1, \ldots, n\}$ such that $\boldsymbol{x}$, $\boldsymbol{y} \in B(\boldsymbol{x}_k, \delta_k)$" in the above proof.