Compactness

Bolzano-Weierstrass

Proposition 1. (Nested intervals) *Let $\{I_n\}$ be a sequence of closed intervals satisfying $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. Then $\cap_{n=1}^{\infty} I_n \neq \varnothing$.*

Proof. Left as exercise. □

Definition 2. *A set $E \subseteq \mathbb{R}^N$ is bounded if and only if there is $R > 0$ such that $E \subseteq B(\mathbf{0}, R)$.*

Exercise 1. Prove that E is bounded if and only if there is an interval $I = [a_1, b_1] \times \cdots \times [a_N, b_N]$ with all a_i, b_i finite such that $E \subseteq I$.

Theorem 3. (Bolzano-Weierstrass) *Every bounded infinite set $A \subseteq \mathbb{R}^n$ has a cluster point.*

Proof. In light of the above exercise, we can assume $A \subseteq I$ for some interval $I = [a_1, b_1] \times \cdots \times [a_N, b_N]$. Divide I into 2^N closed subintervals by bi-secting each interval. Since A is infinite, at least one of these subintervals still contain infinitely many points from A. Denote it by I_1. Do the same thing to I_1 to obtain I_2, and so on.

This way we obtain a sequence of nested intervals $I \supseteq I_1 \supseteq I_2 \cdots$. By the Nested Intervals Theorem we know there is $\boldsymbol{x}_0 \in \cap_{k=1}^{\infty} I_k$. We claim that it is a cluster point of A.

Assume otherwise. Then there is $r > 0$ such that $(B(\boldsymbol{x}_0, r) - \{\boldsymbol{x}_0\}) \cap A = \varnothing$. But since $r > 0$ there is $n > 0$ such that $I_n \subset B(\boldsymbol{x}_0, r)$. Contradiction. □

Exercise 2. From the proof we know that $A \cap I_k \neq \varnothing$ for any k. Can we conclude that $A \cap (\cap_{k=1}^{\infty} I_k) \neq \varnothing$? Justify your answer.

Exercise 3. What is wrong with the following proof?

> Divide I into two closed subintervals $I = J \cup K$. Since A is infinite, at least one of the two intersections $A \cap J$, $A \cap K$ is infinite. Denote it by I_1. Now divide I_1 into two subintervals and repeat the argument, we obtain I_2. Do this again and again we have a sequence of nested intervals $I \supseteq I_1 \supseteq I_2 \supseteq \cdots$. Thanks to Nested Intervals Theorem there is $\boldsymbol{x}_0 \in \cap_{k=1}^{\infty} I_k$ and it is the desired cluster point.

Exercise 4. Let $A = \{\boldsymbol{x}_n\} \subseteq \mathbb{R}^N$ be a sequence of points. Let $\boldsymbol{x}_0 \in \mathbb{R}^N$ be such that there is a subsequence $\boldsymbol{x}_{n_k} \longrightarrow \boldsymbol{x}_0$. Explain why $\boldsymbol{x}_0$ **may not** be a cluster point of A.

Problem 1. Is it possible to prove Theorem 3 using the one dimensional Bolzano-Weierstrass theorem? Explore this possibility and justify your claims.

Definition of compact sets

Definition 4. *A set $E \subseteq \mathbb{R}^N$ is* compact *if and only if every open cover of it has a finite sub-cover. In other words, if there is a collection W of open sets such that*

$$E \subseteq \cup_{A \in W} A \tag{1}$$

then there is $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in W$ such that

$$E \subseteq \cup_{i=1}^{n} A_i. \tag{2}$$

Example 5. Let $E = \{\boldsymbol{x}_0\}$ be just one point. Prove E is compact.

Proof. Let W be any collection of open sets covering E, that is

$$E \subseteq \cup_{A \in W} A. \tag{3}$$

Then by definition of set union:

$$\cup_{A \in W} A := \{\boldsymbol{x} | \exists A \in W, \quad \boldsymbol{x} \in A\} \tag{4}$$

there is one $A_1 \in W$ such that $\boldsymbol{x}_0 \in A_1$. This means $E = \{\boldsymbol{x}_0\} \subseteq A_1$ and we have found our finite sub-cover. □

Exercise 5. Let $E = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$ be a set of finitely many points in $\mathbb{R}^n$. Prove that E is compact.

Example 6. Let $E = \left\{\frac{1}{n} | n \in \mathbb{N}\right\}$. Then E is not compact.

Proof. Consider the following collection of open sets:

$$W := \{E_n | n \in \mathbb{N}\} \text{ with } E_n := \left(\frac{1}{n} - \frac{1}{2n(n+1)}, \frac{1}{n} + \frac{1}{2n(n-1)}\right). \tag{5}$$

Then clearly each E_n is open and $\frac{1}{n} \in E_n$ for every $n \in \mathbb{N}$. Therefore

$$E \subseteq \cup_{n=1}^{\infty} E_n. \tag{6}$$

Now prove by contradiction. Take any finite number of sets from W: $E_{n_1}, \ldots, E_{n_k}$. We show that

$$E \nsubseteq E_{n_1} \cup E_{n_2} \cup \cdots \cup E_{n_k}. \tag{7}$$

Take $n \notin \{n_1, \ldots, n_k\}$. Then for each $l \in \{1, 2, \ldots, k\}$, we have the following two situations:

- $n > n_k$. In this case we have

  $$\frac{1}{n_k} - \frac{1}{n} \geqslant \frac{1}{n_k} - \frac{1}{n_k + 1} = \frac{1}{n_k(n_k+1)} > \frac{1}{2n_k(n_k+1)} \tag{8}$$

  which means

  $$\frac{1}{n} \notin E_{n_k}; \tag{9}$$

- $n < n_k$. Similarly we have

  $$\frac{1}{n} - \frac{1}{n_k} \geqslant \frac{1}{n_k - 1} - \frac{1}{n_k} > \frac{1}{2n_k(n_k-1)} \Longrightarrow \frac{1}{n} \notin E_{n_k}. \tag{10}$$

Thus

$$E \ni \frac{1}{n} \notin E_{n_1} \cup E_{n_2} \cup \cdots \cup E_{n_k} \Longrightarrow E \nsubseteq E_{n_1} \cup E_{n_2} \cup \cdots \cup E_{n_k}. \tag{11}$$

So there is no finite sub-cover and E is not compact. □

Exercise 6. Let $E = \mathbb{N}$. Prove that E is not compact.

Exercise 7. Let $E=(a,b]$ and half-open-half-closed interval in $\mathbb{R}$. Prove that E is not compact.

Example 7. Let $E=\{0\}\cup\left\{\frac{1}{n}|\,n\in\mathbb{N}\right\}$. Prove E is compact.

Proof. Let W be any open cover of E. Then there is $E_0\in W$ such that $0\in E_0$. As E_0 is open, there is $N\in\mathbb{N}$ such that

$$\left(-\frac{1}{N},\frac{1}{N}\right)\subseteq E_0. \tag{12}$$

Now choose E_n such that

$$\frac{1}{n}\in E_n \tag{13}$$

for $n=1,2,\ldots,N$. We claim

$$E\subseteq\cup_{n=0}^{N}E_n. \tag{14}$$

We only need to verify that $\frac{1}{n}\in\cup_{n=0}^{N}E_n$ for all $n>N$. But for such n we have

$$\frac{1}{n}\in\left(-\frac{1}{N},\frac{1}{N}\right)\subseteq E_0. \tag{15}$$

Thus ends the proof. □

Exercise 8. Let $E\subseteq\mathbb{R}^N$. Prove the following.

> E is compact if and only if for any collection W of closed sets, if $E\cap(\cap_{A\in W}A)=\varnothing$, then there are finitely many $A_1,\ldots,A_n\in W$ such that $E\cap(\cap_{k=1}^{n}A_k)=\varnothing$.

Exercise 9. Let $E\subseteq\mathbb{R}^N$ be compact. Let W be an open cover of E. Then there is $r>0$, such that if $B\subseteq A$ is a ball of radius r, then there is one single open set $O\in W$, such that $B\subseteq O$.