More on open and closed sets

Interior, closure, boundary

Most sets in $\mathbb{R}^N$ can be neither open nor closed. These sets can be very complex. Fortunately, for any $A \subseteq \mathbb{R}^N$, there are some open/closed sets closely related to it. These sets are its interior, closure, and boundary.

Exercise 1. Find $A \subseteq \mathbb{R}^2$ that is neither open nor closed. Justify your answer.

Definition 1. *Let $A \subseteq \mathbb{R}^N$. Define*

- **(Interior)** *its interior A^o to be the union of all open sets contained in A:*

$$A^o := \cup_{E \subseteq A, E \text{ open}} E; \tag{1}$$

- **(Closure)** *its closure $\overline{A}$ to be the intersection of all closed sets containing A:*

$$\overline{E} := \cap_{E \supseteq A, E \text{ closed}} E; \tag{2}$$

- **(Boundary)** *its boundary ∂A to be*

$$\partial A := \overline{A} - A^o. \tag{3}$$

Exercise 2. Let $A \subseteq \mathbb{R}^N$. Prove the following:

a) A^o is the largest open set contained in A, in the sense that any $U \subseteq A$, if U is open, then $U \subseteq A^o$;

b) $\bar{A}$ is the smallest closed set containing A.

Exercise 3. Let $A \subseteq \mathbb{R}^N$. Prove that A is open if and only if A equals the union of all open balls contained in it.

Exercise 4. Prove the following:

a) A is open if and only if $A = A^o$;

b) A is closed if and only if $A = \bar{A}$.

Example 2. Find the interior, closure, boundary of the following sets.

a) $A := \{(x, y) \in \mathbb{R}^2 | \, x < y\}$;

b) $A \subseteq \mathbb{R}^N$ consisting of finitely many points.

c) $A := \left\{\left(\frac{1}{n}, \frac{1}{m}\right) | \, n, m \in \mathbb{N}\right\} \subseteq \mathbb{R}^2$.

d) $A := \mathbb{Q} \times \mathbb{Q} \subset \mathbb{R}^2$.

e) A is a hyperplane.

Solution. We solve part d) and leave the rest as exercises.

First we claim that the interior A^o is empty. By definition, all we need to show is that for any U open, $U \not\subseteq A$. Take any U open. By definition of open sets there is a ball $B(\boldsymbol{x}_0, r) \subseteq A$. Now if $\boldsymbol{x}_0 \notin \mathbb{Q} \times \mathbb{Q}$, we already have $U \not\subseteq A$; Otherwise take $0 < r_1 < r$ such that $r_1^2 \notin \mathbb{Q}$. Thus any vector $\boldsymbol{y} \in S(\boldsymbol{0}, r_1) \notin \mathbb{Q} \times \mathbb{Q}$ and consequently $\boldsymbol{x}_0 + \boldsymbol{y} \notin \mathbb{Q} \times \mathbb{Q} = A$.

Next we claim that the closure of A is $\mathbb{R}^2$. To show this we need to show that any closed set E satisfying $A \subseteq E$, we must have $E = \mathbb{R}^2$, or equivalently $E^c = \varnothing$. Assume otherwise. As E^c is open by definition of closed sets, there is a ball $B(\boldsymbol{x}_0, r) \subseteq E^c \subseteq A^c$ for some $r > 0$. By density of A in $\mathbb{R}^2$ we see that $B(\boldsymbol{x}_0, r) \cap A \neq \varnothing$. Contradiction.

Finally we have $\partial A = \bar{A} - A^o = \mathbb{R}^2$.

Exercise 5. Solve a), b), d), e).

Lemma 3. *Let $A \subseteq B \subseteq \mathbb{R}^N$. Then $A^o \subseteq B^o$, $\overline{A} \subseteq \overline{B}$.*

Proof. We prove the second claim and leave the first as exercise.

Notice that $\bar{B}$ is closed, and $A \subseteq B \subseteq \bar{B}$. Thus by definition $\bar{A} \subseteq \bar{B}$. □

Proposition 4. (Properties of interior) *Let $A \subseteq \mathbb{R}^N$. Then*

a) A^o *is open;*

b) $A^o \subseteq A$;

c) $(A^o)^o = A^o$;

d) $(A \cap B)^o = A^o \cap B^o$;

Proof. a), b), c) are trivial and left as exercises. We prove d) here. Recall that to prove equality of two sets, we need to prove one is a subset of the other and vice versa.

- $A^o \cap B^o \subseteq (A \cap B)^o$. As $A^o \subseteq A, B^o \subseteq B$, we have $A^o \cap B^o \subseteq A \cap B$; Furthermore as A^o, B^o are open, we have $A^o \cap B^o$ is open. By definition of interior we have

$$A^o \cap B^o \subseteq (A \cap B)^o; \tag{4}$$

- $(A \cap B)^o \subseteq A^o \cap B^o$. Since $(A \cap B) \subseteq A$, we have $(A \cap B)^o \subseteq A^o$. Similarly $(A \cap B)^o \subseteq B^o$. Therefore $(A \cap B)^o \subseteq A^o \cap B^o$. □

Proposition 5. (Properties of closure) *Let $A \subseteq \mathbb{R}^N$. Then*

a) $\bar{A}$ *is closed;*

b) $A \subseteq \bar{A}$;

c) Closure of closure of A equals closure of A: $\overline{(\bar{A})} = \bar{A}$.

d) $\overline{(A \cup B)} = \bar{A} \cup \bar{B}$.

Proof. Left as exercises. □

Exercise 6. Find two sets A, B such that $\overline{A \cap B} \subsetneq \overline{A} \cap \overline{B}$.

Proposition 6. (Properties of boundary) *Let $A \subseteq \mathbb{R}^N$. Then*

a) $\partial A = \{\boldsymbol{x} | \, \forall r > 0, B(\boldsymbol{x}, r) \cap A \neq \varnothing$ *and* $B(\boldsymbol{x}, r) \cap A^c \neq \varnothing\}$;

b) ∂A *is closed;*

c) $\partial(\partial A) \subseteq \partial A$.

d) Let $A, B \subseteq \mathbb{R}^N$ *then*

$$\partial(A \cup B) \subseteq (\partial A) \cup (\partial B); \qquad \partial(A \cap B) \supseteq (\partial A) \cap (\partial B). \tag{5}$$

Proof. Left as exercises. □

Exercise 7. For each relation in d), find an example where "=" holds and an example where "⊂" holds.

Exercise 8. Critique the following claim:

Let $A \subseteq \mathbb{R}^N$. Then $\partial A = \{\boldsymbol{x} \in \mathbb{R}^n | \operatorname{dist}(\boldsymbol{x}, A) = \operatorname{dist}(\boldsymbol{x}, A^c) = 0\}$.

If you think it is true, prove; Otherwise provide a counterexample.

Exercise 9. Let $E \subseteq \mathbb{R}^N$ be convex. Prove that if $\boldsymbol{x} \in \partial E$, then $\boldsymbol{x} \in \partial((\overline{E})^c)$. Find a non-convex set S for which this claim does not hold.

Cluster point

Definition 7. (Cluster point) *Let* $A \subseteq \mathbb{R}^N$. $\boldsymbol{x}_0$ *is a cluster point* A *if and only if for any open set* U *containing* $\boldsymbol{x}_0$, $A \cap (U - \{\boldsymbol{x}_0\}) \neq \varnothing$.

Remark 8. Recall our discussion on limit of functions. Now we can say this can only be discussed at cluster points of the domain of f.

Exercise 10. Let $A \subseteq \mathbb{R}^N$ be open. Then

a) Any $\boldsymbol{x} \in A$ is a cluster point of A;

b) Find an open set such that there is $\boldsymbol{x} \notin A$ but is a cluster point of A.

Exercise 11. Find a closed set $A \subseteq \mathbb{R}^N$ satisfying each of the following (not simultaneously!)

a) A has no cluster point;

b) Any $\boldsymbol{x} \in A$ is a cluster point of A.

Example 9. $\mathbb{N}$ has no cluster point in $\mathbb{R}$.

Proof. Take any $x \in \mathbb{R}$. There are two cases:

1. $x \in \mathbb{N}$. Take $r = 1/2$. Then $\mathbb{N} \cap (B(x, r) - \{x\}) = \varnothing$;

2. $x \notin \mathbb{N}$. Let $m \in \mathbb{N}$ be such that $m < x < m + 1$. Take $r = \frac{1}{2} \min(|x - m|, |x - m - 1|)$, then we again have $\mathbb{N} \cap (B(x, r) - \{x\}) = \varnothing$. □

Exercise 12. Find the cluster point(s) for the set $S := \{1/n | n \in \mathbb{N}\} \subset \mathbb{R}$. Justify your answer.

Exercise 13. Find the cluster point(s) for the set $E := \mathbb{Q} \times (\mathbb{R} - \mathbb{Q}) \subset \mathbb{R}^2$, that is $E := \{(x, y) | x \in \mathbb{Q}, y \notin \mathbb{Q}\}$.

Example 10. Let $\boldsymbol{x}_0 \in \mathbb{R}^N$ and $r > 0$. Then the set of cluster points for the open ball $B(\boldsymbol{x}, r)$ is its closure $\overline{B(\boldsymbol{x}, r)}$.

Proposition 11. *Let $\boldsymbol{x}\in\mathbb{R}^N$ and $A\subseteq\mathbb{R}^N$. The following are equivalent.*

a) *$\boldsymbol{x}$ is a cluster point of A;*

b) *$\boldsymbol{x}\in\overline{A-\{\boldsymbol{x}\}}$;*

c) *$\operatorname{dist}(\boldsymbol{x},A-\{\boldsymbol{x}\})=0$;*

d) *For any open set U containing $\boldsymbol{x}_0$, $A\cap(U-\{\boldsymbol{x}_0\})$ has infinitely many points.*

Proof. We only prove a)$\Longrightarrow$d) here and leave the rest, which are much easier, as exercises.

Let $\boldsymbol{x}_0$ be a cluster point of A. We will construct a sequence $\{\boldsymbol{x}_n\}\subseteq A$, $\boldsymbol{x}_n\neq\boldsymbol{x}_0$ for all n, such that for any open set U containing $\boldsymbol{x}_0$, there is $K\in\mathbb{N}$ that for all $n>K$, $\boldsymbol{x}_n\in U$.

First consider the open ball $B(\boldsymbol{x}_0,\ 1/2)$. Since $\boldsymbol{x}$ is a cluster point of A, there is a point in $A\cap B(\boldsymbol{x}_0,\ 1/2)$. Call it $\boldsymbol{x}_1$. Now there must be $\boldsymbol{x}_0\neq\boldsymbol{x}_2\in B\left(\boldsymbol{x}_0,\ \frac{\|\boldsymbol{x}_1-\boldsymbol{x}_0\|}{2}\right)\cap A$. Next we find $\boldsymbol{x}_0\neq\boldsymbol{x}_3\in B\left(\boldsymbol{x}_0,\frac{\|\boldsymbol{x}_2-\boldsymbol{x}_0\|}{2}\right)\cap A$, and so on.

Now observe that $\|\boldsymbol{x}_n-\boldsymbol{x}_0\|<2^{-n}$. For any open set U containing $\boldsymbol{x}_0$, there is $r>0$ such that $B(\boldsymbol{x}_0,r)\subseteq U$. Now choose $K\in\mathbb{N}$ such that $K>-\log_2 r$. We have, for all $n>K$,

$$\|\boldsymbol{x}_n-\boldsymbol{x}_0\|<2^{-K}<r\Longrightarrow\boldsymbol{x}_n\in B(\boldsymbol{x}_0,r)\subseteq U. \tag{6}$$

Thus ends the proof. □

Exercise 14. Complete the proof of the proposition.

Definition 12. (Isolated point) *If $\boldsymbol{x}\in A$ is not a cluster point of A, we say $\boldsymbol{x}$ is an isolated point of A.*

Exercise 15. Prove that $\boldsymbol{x}$ is a cluster point of A if and only if it is not an isolated point of A.

Proposition 13. *Let $\boldsymbol{x}\in\mathbb{R}^N$ and $A\subseteq\mathbb{R}^N$. Then $\boldsymbol{x}$ is an isolated point of A if and only if $\boldsymbol{x}\in A$ but $\operatorname{dist}(\boldsymbol{x},A-\{\boldsymbol{x}\})>0$.*

Exercise 16. Find the cluster and isolated points of the following sets. Justify your answers.

a) $A=\{(x,y)|\,|x|+|y|\leqslant 1];$

b) $B=\{(x,y)|\,x\geqslant 0\};$

c) $C=\{(x,y)|\,x^2+y^2<1\};$

d) $D=\{(1/m,1/n)|\,m,n\in\mathbb{N}\};$

e) $E=\{(x,y)|\,x^2<1\}\cup\{(x,y)|\,y^2>1\}.$

Lemma 14. *Let $A\subseteq\mathbb{R}^N$. Then*

$$A\text{ is closed}\Longleftrightarrow A\text{ contains all its cluster points.} \tag{7}$$

Proof.

$\Longrightarrow$. Assume there is $\boldsymbol{x}_0 \notin E$ that is a cluster point of E. Then for each m, there is $\boldsymbol{x}_m \in B(\boldsymbol{x}_0, 1/m) \cap E$. On the other hand, as E is closed, E^c is open, which means there is $\varepsilon_0 > 0$ such that $B(\boldsymbol{x}_0, \varepsilon_0) \cap E = \varnothing$. Taking $m > \varepsilon_0^{-1}$ leads to contradiction.

$\Longleftarrow$. Assume that A is not closed. Then by definition A^c is not open. This means there is $\boldsymbol{x}_0 \in A^c$ such that for any open set $U \ni \boldsymbol{x}_0$, $U \not\subseteq A^c$. It follows that $U \cap A \neq \varnothing$. Since $\boldsymbol{x}_0 \notin A$, there must be $\boldsymbol{x} \neq \boldsymbol{x}_0$ inside $U \cap A$. By definition $\boldsymbol{x}_0$ is a cluster point of A. But by assumption A does not contain $\boldsymbol{x}_0$. Contradiction. □

Problem 1. Let $A \subseteq \mathbb{R}^N$. Prove $(\overline{A^c})^c = A^o$, that is we can represent interior using closure and complement only. Can you find a similar equality for $\bar{A}$?

Problem 2. Let $A \subseteq \mathbb{R}^N$. Prove that $\partial(\overline{A}) \subseteq \partial A, \partial(A^o) \subseteq \partial A$. Find counterexamples to show that $\subseteq$ cannot be replaced by $=$.

Problem 3. Let C be convex and nonempty. Prove

$$(\overline{C})^o = C^o, \qquad \overline{C^o} = \overline{C}. \tag{8}$$

Do these relations hold for arbitrary set A? Justify your claims.

Problem 4. ([?]) Let $A \subseteq \mathbb{R}^N$ be nonempty. Let W be the collection of sets obtained from E by applying ${}^c, {}^o, {}^-$ finitely many times in any order. Prove that W has at most 14 elements.

Problem 5. Let $\{\boldsymbol{x}_n\} \subset \mathbb{R}^N$ be a sequence. Let $A = \{\boldsymbol{x} | \ \boldsymbol{x}$ is a cluster point of the set $\{\boldsymbol{x}_n\}\}$ and $B = \{\boldsymbol{x} | \ \boldsymbol{x} = \lim_{k \longrightarrow \infty} \boldsymbol{x}_{n_k}$ for some subsequence $\{\boldsymbol{x}_{n_k}\}\}$. Explore the relation between A and B. Justify your answer.

Problem 6. Let I be an interval in $\mathbb{R}$. Let $f: \mathbb{R} \mapsto \mathbb{R}$ be continuous on I. Find a counterexample for each of the following claims:

a) I is closed $\Longrightarrow$ $f(I)$ is closed;

b) I is closed $\Longrightarrow$ $f(I)$ is bounded;

c) I is open $\Longrightarrow$ $f(I)$ is open;

d) I is bounded $\Longrightarrow$ $f(I)$ is bounded;

e) I is bounded and open $\Longrightarrow$ $\max f(I)$ does not exist;

Problem 7. Let $f: \mathbb{R} \mapsto \mathbb{R}$ be bounded. Assume that its graph $\{(x, f(x)) | \ x \in \mathbb{R}\}$ is a closed set in $\mathbb{R}^2$. Prove that f is continuous. Can you generalize this to $\mathbb{R}^N$?