Analytic geometry in $\mathbb{R}^N$

In this final section we will introduce basic geometric shapes in $\mathbb{R}^N$ and study the change they go through during linear transformations.

Geometric objects in $\mathbb{R}^N$

Intervals

Definition 1. (Intervals) *Let $a_1, \ldots, a_N, b_1, \ldots, b_N \in \mathbb{R}$ satisfy $a_i \leqslant b_i$. Then*

$$I := \{\boldsymbol{x} \in \mathbb{R}^N | \ x_i \in (a_i, b_i) \text{ for all } i = 1, 2, \ldots, N\} \tag{1}$$

is called a open interval *in $\mathbb{R}^N$, and*

$$I := \{\boldsymbol{x} \in \mathbb{R}^N | \ x_i \in [a_i, b_i] \text{ for all } i = 1, 2, \ldots, N\} \tag{2}$$

is called a closed interval *in $\mathbb{R}^N$.*

Balls

Definition 2. (Ball and sphere) *Let $\boldsymbol{x}_0 \in \mathbb{R}^N$ and $r > 0$. Then*

$$B(\boldsymbol{x}_0, r) := \{\boldsymbol{x} \in \mathbb{R}^N | \ \|\boldsymbol{x} - \boldsymbol{x}_0\| < r\} \tag{3}$$

is called the open ball *of center $\boldsymbol{x}_0$ and radius r;*

$$\overline{B(\boldsymbol{x}_0, r)} := \{\boldsymbol{x} \in \mathbb{R}^N | \ \|\boldsymbol{x} - \boldsymbol{x}_0\| \leqslant r\} \tag{4}$$

is called the closed ball *of center $\boldsymbol{x}_0$ and radius r. Their difference*

$$S(\boldsymbol{x}_0, r) := \{\boldsymbol{x} \in \mathbb{R}^N | \ \|\boldsymbol{x} - \boldsymbol{x}_0\| = r\} \tag{5}$$

is called the sphere *of center $\boldsymbol{x}_0$ and radius r.*

Exercise 1. Prove that when $N = 1$, the open and closed balls reduce to open and closed intervals. What does the sphere reduce to when $N = 1$?

Exercise 2. (Nested balls) Try to formulate and prove "nested balls theorem".

Hyperplanes and half spaces

Definition 3. (Hyperplane) *A hyperplane in $\mathbb{R}^N$ is a set of the form*

$$P := \{\boldsymbol{x} | \ \boldsymbol{a} \cdot \boldsymbol{x} = b\} \tag{6}$$

for some vector $\boldsymbol{a} \in \mathbb{R}^N$, $\boldsymbol{a} \neq \boldsymbol{0}$, and some $b \in \mathbb{R}$.

Lemma 4. *Let P be a hyperplane defined through $P := \{\boldsymbol{x} | \ \boldsymbol{a} \cdot \boldsymbol{x} = b\}$. Then the vector $\boldsymbol{a}$ is in fact the normal vector of the hyperplane, that is*

$$\forall \boldsymbol{x}, \boldsymbol{y} \in P, \qquad \boldsymbol{a} \perp (\boldsymbol{x} - \boldsymbol{y}). \tag{7}$$

Proof. Take any $\boldsymbol{x}, \boldsymbol{y} \in P$, we have

$$\boldsymbol{a} \cdot (\boldsymbol{x} - \boldsymbol{y}) = \boldsymbol{a} \cdot \boldsymbol{x} - \boldsymbol{a} \cdot \boldsymbol{y} = b - b = 0. \tag{8}$$

Thus ends the proof. □

Exercise 3. Let P be a hyperplane and let $\boldsymbol{x} \notin P$. Assume there is $\boldsymbol{y}_0 \in P$ such that

$$\forall \boldsymbol{y} \in P, \qquad \|\boldsymbol{x} - \boldsymbol{y}_0\| \leqslant \|\boldsymbol{x} - \boldsymbol{y}\|, \tag{9}$$

then $\boldsymbol{x} - \boldsymbol{y}_0 \perp P$. Is the converse claim true? Justify your answer.

Definition 5. (Half space) *An "open half space" in $\mathbb{R}^N$ is a set E of the form $E = \{\boldsymbol{x} | \boldsymbol{a} \cdot \boldsymbol{x} > b\}$ for some vector $\boldsymbol{a} \in \mathbb{R}^N$, $\boldsymbol{a} \neq \boldsymbol{0}$, and some $b \in \mathbb{R}$. The sets defined through $\{\boldsymbol{x} | \boldsymbol{a} \cdot \boldsymbol{x} \geqslant b\}$ are called "closed half spaces".*

Remark 6. We see that open/closed intervals are intersections of $2n$ open/closed half-spaces.

Exercise 4. Explain why there is no need to consider $\{\boldsymbol{x} | \boldsymbol{a} \cdot \boldsymbol{x} < b\}$.

Convex sets

Definition 7. *A set $E \subseteq \mathbb{R}^N$ is said to be convex if and only if*

$$\forall \boldsymbol{x}, \boldsymbol{y} \in E \quad \forall t \in [0, 1] \qquad t\,\boldsymbol{x} + (1 - t)\,\boldsymbol{y} \in E. \tag{10}$$

Example 8. Both open and closed balls are convex, while the sphere is not convex.

Proof. We prove the claim for open balls and leave the other cases as exercises.

Take any two $\boldsymbol{x}, \boldsymbol{y} \in B(\boldsymbol{x}_0, r)$. Then by definition

$$\|\boldsymbol{x} - \boldsymbol{x}_0\|, \|\boldsymbol{y} - \boldsymbol{y}_0\| < r. \tag{11}$$

Triangle's inequality then gives, for any $t \in [0, 1]$

$$\|(t\,\boldsymbol{x} + (1 - t)\,\boldsymbol{y}) - \boldsymbol{x}_0\| = \|t\,(\boldsymbol{x} - \boldsymbol{x}_0) + (1 - t)\,(\boldsymbol{y} - \boldsymbol{x}_0)\| \leqslant \|t\,(\boldsymbol{x} - \boldsymbol{x}_0)\| + \|(1 - t)\,(\boldsymbol{y} - \boldsymbol{x}_0)\| \tag{12}$$

which is <1. □

Exercise 5. Prove that half spaces are convex.

Exercise 6. Make a statement about convexity of open and closed intervals in $\mathbb{R}^N$, and prove your claims.

Exercise 7. Let $E \subset \mathbb{R}^N$ be defined by

$$E := \{\boldsymbol{x} \in \mathbb{R}^N | \|\boldsymbol{x}\| < 1\} \cup \{(1, 0, \ldots, 0)\}. \tag{13}$$

Is E convex? Justify your answer.

Exercise 8. Let $\boldsymbol{x}_0 \in \mathbb{R}^N$, $r > 0$. Let $S \subseteq S(\boldsymbol{x}_0, r)$ be any subset of the sphere. Define

$$E := B(\boldsymbol{x}_0, r) \cup S. \tag{14}$$

Is E convex? Justify your answer.

Lemma 9. *Let W be a collection of half spaces. Then their intersection*

$$E := \cap_{P \in W} P \tag{15}$$

is convex.

Proof. Take any $\boldsymbol{x}, \boldsymbol{y} \in E$. Then $\boldsymbol{x}, \boldsymbol{y} \in P$ for any $P \in W$. Let $t \in [0, 1]$ be arbitrary. Since P is convex, we have $t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y} \in P$. Therefore E is convex. □

Exercise 9. Let W be a collection of convex sets. Prove their intersection is still convex.

Exercise 10. Recall that a function $f: \mathbb{R} \mapsto \mathbb{R}$ is convex if and only if for all $x, y \in \mathbb{R}$ and all $t \in [0, 1]$,

$$f(t\,x + (1-t)\,y) \geqslant t\,f(x) + (1-t)\,f(y). \tag{16}$$

Define the epigraph of a function to be

$$\mathrm{epi}(f) := \{(x, y) \in \mathbb{R}^2 |\, y \geqslant f(x)\}. \tag{17}$$

Prove: $f: \mathbb{R} \mapsto \mathbb{R}$ is convex if and only if epi(f) is a convex set in $\mathbb{R}^2$.

Exercise 11. Generalize the above definitions and claim to $f: \mathbb{R}^n \mapsto \mathbb{R}$. Can you still prove the claim?

Linear transformation of sets in $\mathbb{R}^n$

General theory

Theorem 10. *Let E be a hyperplane in $\mathbb{R}^n$ and $\boldsymbol{f}$ a linear transformation on $\mathbb{R}^n$. Then $\boldsymbol{f}(E) \subseteq P$ wher P is a hyperplane.*

Proof. Let the equation for E be $\boldsymbol{a} \cdot \boldsymbol{x} = b$. We try to find another vector $\tilde{\boldsymbol{a}} \in \mathbb{R}^n$ and some $b \in \mathbb{R}$ such that $\tilde{\boldsymbol{a}} \cdot \boldsymbol{f}(\boldsymbol{x}) = \tilde{b}$. As

$$\tilde{\boldsymbol{a}} \cdot \boldsymbol{f}(\boldsymbol{x}) = (\tilde{\boldsymbol{a}})^T A\,\boldsymbol{x} = (A^T \tilde{\boldsymbol{a}}) \cdot \boldsymbol{x}, \tag{18}$$

we see that if we take $\tilde{\boldsymbol{a}} = A^{-T} \boldsymbol{a}$, then $(A^T \tilde{\boldsymbol{a}}) \cdot \boldsymbol{x} = (A^T A^{-T} \boldsymbol{a}) \cdot \boldsymbol{x} = \boldsymbol{a} \cdot \boldsymbol{x} = b$. □

Corollary 11. *Let E_1, E_2 be two hyperplanes that are parallel, then $\boldsymbol{f}(E_1), \boldsymbol{f}(E_2)$ are still parallel.*

Thus if $I \subseteq \mathbb{R}^n$ is an interval, $\boldsymbol{f}(I)$ is a parallelopiped.

Theorem 12. *Let E be a convex set in $\mathbb{R}^n$ and $\boldsymbol{f}$ a linear transformation on $\mathbb{R}^n$. Then $\boldsymbol{f}(E)$ is still convex.*

Proof. Take any $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \boldsymbol{f}(E)$ and $t \in [0, 1]$. Let $\boldsymbol{x}_1, \boldsymbol{x}_2$ be the preimages of $\boldsymbol{y}_1, \boldsymbol{y}_2$. We have

$$t\,\boldsymbol{y}_1 + (1-t)\,\boldsymbol{y}_2 = t\,\boldsymbol{f}(\boldsymbol{x}_1) + (1-t)\,\boldsymbol{f}(\boldsymbol{x}_2) = \boldsymbol{f}(t\,\boldsymbol{x}_1 + (1-t)\,\boldsymbol{x}_2) \in \boldsymbol{f}(E). \tag{19}$$

Note that in the last step we have used the convexity of E. □

Effect of special transforms

Example 13. Let $\boldsymbol{f}$ be a transformation with an orthogonal representation O. Then $\boldsymbol{f}(E)$ is identical to E for any set $E \subseteq \mathbb{R}^n$.

To see this we recall that $\|O\,\boldsymbol{x}\| = \|\boldsymbol{x}\|$ for any $\boldsymbol{x} \in \mathbb{R}^n$. It then follows that

$$\mathrm{dist}(O\,\boldsymbol{x}, O\,\boldsymbol{y}) = \mathrm{dist}(\boldsymbol{x}, \boldsymbol{y}) \tag{20}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in E$.