MATH 118 WINTER 2015 LECTURE 46 (APR. 8, 2015)

Final Review I: Optimization & Convexity

- Theory.
  - Mathematical optimization problem:

$$\min/\max \ f(x) \qquad \text{subject to } a \leqslant x \leqslant b \quad (\text{or } a < x < b, a < x \leqslant b, a \leqslant x < b) \tag{1}$$

    Make sure you know the equivalence of maximization and minimization problems.

  - Global and local minimizers.
    - Global minimizer:

$$\forall x \in [a, b], \qquad f(x_0) \leqslant f(x). \tag{2}$$

    - Local minimizer:

$$\exists \delta > 0, \qquad \forall x \in [a, b] \cap (x_0 - \delta, x_0 + \delta), \qquad f(x_0) \leqslant f(x). \tag{3}$$

    - Interior local minimizer:

$$x_0 \in (a, b), \quad \exists \delta > 0, \quad \forall x \in [a, b] \cap (x_0 - \delta, x_0 + \delta), \qquad f(x_0) \leqslant f(x). \tag{4}$$

    - Relations:

$$x_0 \text{ is a global minimizer} \Longrightarrow x_0 \text{ is a local minimizer}; \tag{5}$$

$$x_0 \text{ is an interior local minimizer} \Longrightarrow x_0 \text{ is a local minimizer}. \tag{6}$$

      Exercise 1. Give counter-examples to the $\Longleftarrow$ direction in (5) and (6).

  - Solving optimization problems.
    - Basic strategy: Assume that f is differentiable in (a, b).
      1. Find all interior local minimizers;
      2. Compare values at these interior local minimizers, together with values at a, b.

      Exercise 2. What if f is differentiable in $(a, c) \cup (c, b)$ for some $c \in (a, b)$ but not differentiable at c?

    - Finding all interior local minimizers.
      - Candidates for interior local minimizers.

        THEOREM. *If f is differentiable in (a, b), then $f' = 0$ at all interior local minimizers.*

        Exercise 3. Can $f' = 0$ at points other than interior local minimizers?

      - Which of these points are interior local minimizers?
        Let $x_0 \in (a, b)$ satisfy $f'(x_0) = 0$.
        - Criterion 1. If there is $\delta > 0$ such that $f'(x) \leqslant 0$ for $x \in (x_0 - \delta, x_0)$ and $f'(x) \geqslant 0$ for $x \in (x_0, x_0 + \delta)$, then x_0 is an interior local minimizer.

          Exercise 4. Show that this criterion is not necessary for x_0 to be an interior local minimizer.

- Criterion 2. If $f''(x_0) > 0$ then x_0 is an interior local minimizer.

  Exercise 5. Show that this criterion is not necessary.

  Make sure you know the corresponding criteria for maximizers.

- Examples.

Example 1. Solve

$$\max/\min f(x) = \frac{x^3}{3} - 2x^2 + 3x + 1 \quad s.t. \quad 0 \leqslant x \leqslant 4. \tag{7}$$

Solution. We have

$$f'(x) = x^2 - 4x + 3 \tag{8}$$

So $f'(x) = 0 \Longrightarrow x_{1,2} = 1, 3$. Now compare

$$f(0) = 1, \quad f(1) = \frac{7}{3}, \quad f(3) = 1, \quad f(4) = \frac{7}{3}. \tag{9}$$

Therefore the global minimizers are $0, 3$, global maximizers are $1, 4$.

Example 2. Find all local minimizers of $f(x) = 2\sin x + \cos 2x$ for $-\infty < x < \infty$.

Solution. We have

$$f'(x) = 2\cos x - 2\sin 2x = 2(2\sin x - 1)\cos x. \tag{10}$$

Setting $f'(x) = 0$ we have $x = 2k\pi + \frac{\pi}{6}, 2k\pi + \frac{5\pi}{6}, k\pi + \frac{\pi}{2}$, $k \in \mathbb{Z}$. As $f(x)$ is periodic with period 2π, we only need to test the points $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}$. To see which of them are local minimizers we calculate

$$f''(x) = -2\sin x - 4\cos 2x \tag{11}$$

and further

$$f''\left(\frac{\pi}{6}\right) = -3, \quad f''\left(\frac{5\pi}{6}\right) = -3, \quad f''\left(\frac{\pi}{2}\right) = 2, \quad f''\left(\frac{3\pi}{2}\right) = 6. \tag{12}$$

Thus we see that the local minimizers are $k\pi + \frac{\pi}{2}$, $k \in \mathbb{Z}$.

Example 3. Find all local minimizers and maximizers to

$$f(x) = (x-1)\,|x|^{2/3} \quad \text{over} \quad -1 \leqslant x \leqslant 1. \tag{13}$$

Solution. We notice that $f(x)$ is not differentiable at $x = 0$. At other points we have

$$f'(x) = \frac{5x - 2}{3x^{1/3}}. \tag{14}$$

Thus $f'(x) = 0 \Longrightarrow x_0 = \frac{2}{5}$. Thus we have four candidates for local minimizers/maximizers: $-1, 0, \frac{2}{5}, 1$.

- -1. We have $f'(x) > 0$ for $x \in (-1, 0)$. Therefore -1 is a local minimizer.
- 0. We have $f'(x) > 0$ for $x \in (-1, 0)$ and $f'(x) < 0$ for $x \in \left(0, \frac{2}{5}\right)$. Therefore 0 is a local maximizer.
- $\frac{2}{5}$. We have $f'(x) < 0$ for $x \in \left(0, \frac{2}{5}\right)$ and $f'(x) > 0$ for $x \in \left(\frac{2}{5}, 1\right)$. Therefore $\frac{2}{5}$ is a local minimizer.

- 1. We have $f'(x) > 0$ for $x \in \left(\frac{2}{5}, 1\right)$. Therefore 1 is a local maximizer.

- Convexity.
  - Definition.
    - A function $f: [a, b] \mapsto \mathbb{R}$ is convex if and only if

$$\forall x, y \in [a, b], \quad \forall \lambda \in [0, 1], \qquad f(\lambda\, x + (1 - \lambda)\, y) \leqslant \lambda\, f(x) + (1 - \lambda)\, f(y). \tag{15}$$

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  - Properties.
    - f is convex on $[a, b] \Longleftrightarrow \forall x_1, \ldots, x_n \in [a, b], \forall \lambda_1, \ldots, \lambda_n \geqslant 0, \lambda_1 + \cdots + \lambda_n = 1,$

$$f(\lambda_1\, x_1 + \cdots + \lambda_n\, x_n) \leqslant \lambda_1\, f(x_1) + \cdots + \lambda_n\, f(x_n). \tag{17}$$

    - f is convex on $[a, b] \Longleftrightarrow \forall a \leqslant x < y < z \leqslant b$, $\frac{f(z) - f(y)}{z - y} \geqslant \frac{f(y) - f(x)}{y - x} \Longleftrightarrow \forall a \leqslant x < y < z \leqslant b$, $\frac{f(z) - f(x)}{z - x} \geqslant \frac{f(y) - f(x)}{y - x}$.
    - f is convex on $[a, b]$ then f is continuous on (a, b).
  - Checking convexity.
    - If f is continuous on $[a, b]$ and differentiable on (a, b), then f is convex on $[a, b]$ if and only if $f'(x)$ is increasing.
    - If f is continuous on $[a, b]$ and twice differentiable on (a, b), then f is convex on $[a, b]$ if and only if $f''(x) \geqslant 0$ on (a, b).