MATH 118 WINTER 2015 LECTURE 41 (MAR. 26, 2015)

- Area of a plane region.
  - $a \leqslant x \leqslant b$, $g(x) \leqslant y \leqslant f(x)$:
$$A = \int_a^b [f(x) - g(x)]\,\mathrm{d}x. \tag{1}$$
  - $c \leqslant y \leqslant d$, $\psi(y) \leqslant x \leqslant \varphi(y)$:
$$A = \int_c^d [\varphi(y) - \psi(y)]\,\mathrm{d}y. \tag{2}$$
  - Parametrized curve $(x(t), y(t))$, $a \leqslant t \leqslant b$, $x(a) = x(b)$, $y(a) = y(b)$. $(x(t), y(t))$ traces the boundary counter-clockwisely as t increases from a to b.
$$A = -\int_a^b y(t)\,x'(t)\,\mathrm{d}t = \int_a^b x(t)\,y'(t)\,\mathrm{d}t = \frac{1}{2}\int_a^b [x(t)\,y'(t) - y(t)\,x'(t)]\,\mathrm{d}t. \tag{3}$$

Example 1. Let $(r(t), \theta(t))$ be the polar coordinate representation of a closed curve in the plane. Further assume that as t increases from a to b, $(r(t), \theta(t))$ traces the boundary counter-clockwisely. Find the formula for its area.

Solution. Substituting $x = r\cos\theta$, $y = r\sin\theta$ into $A = \frac{1}{2}\int_a^b [x(t)\,y'(t) - y(t)\,x'(t)]\,\mathrm{d}t$ we obtain
$$A = \frac{1}{2}\int_a^b r^2(t)\,\theta'(t)\,\mathrm{d}t. \tag{4}$$

Remark 2. When the curve is given by $r = r(\theta)$, the formula for the area is
$$A = \frac{1}{2}\int_a^b r^2(\theta)\,\mathrm{d}\theta. \tag{5}$$

Exercise 1. Calculate the area enclosed by $r^2 = \cos 2\theta$. (Ans:[1])

Exercise 2. Calculate the area enclosed by $r = 1 + \cos\theta$. (Ans:[2])

- Volume.
  - Basic idea.

    If the area of the cross-section of the 3D shape and the plane parellel to y-z plane, passing $(x, 0, 0)$ is $A(x)$, then the volume of the shape between $x = a$ and $x = b$ is given by
$$\int_a^b A(x)\,\mathrm{d}x. \tag{6}$$
  - Examples.

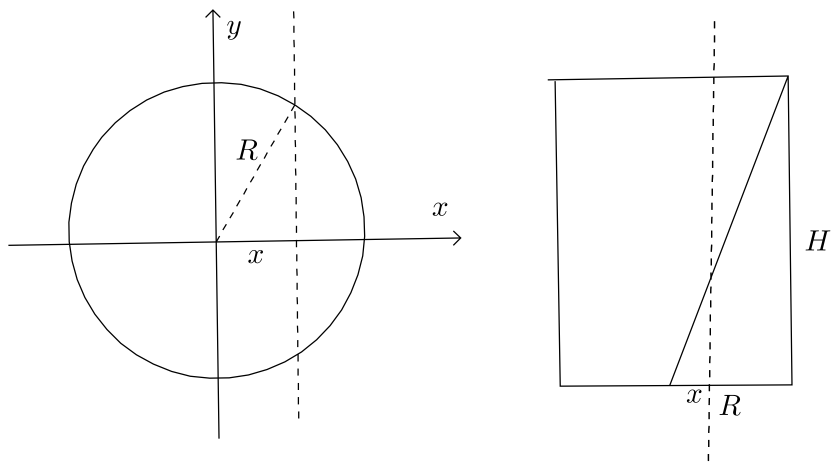
Example 3. Calculate the volume of the ball enclosed by $x^2 + y^2 + z^2 = R^2$.

Solution. If we cut the ball with a plane parallel to y-z and passing $(x, 0, 0)$, the cross-section is a disk with radius $r = \sqrt{R^2 - x^2}$. Therefore the volume is given by
$$V = \int_{-R}^{R} \pi r^2\,\mathrm{d}x = \frac{4\pi}{3} R^3. \tag{7}$$

1. 1.
2. $\frac{3\pi}{2}$.

Example 4. Consider the cylinder enclosed by $x^2+y^2=R^2$, $z=0$, $z=H$. Now cut it by the plane passing the y axis and $(R,0,H)$. Calculate the volume of the smaller part.

Solution. First we realize the the cross-sections are rectangles. With base and height as shown in the following (The left is the base disk of the cylinder, the right is when we look at it from the $-y$ direction).



Thus the base of the rectangle is $2\sqrt{R^2-x^2}$ and the height is $x\,\frac{H}{R}$. So we calculate the volume as

$$V=\int_0^R 2\sqrt{R^2-x^2}\,x\,\frac{H}{R}\,\mathrm{d}x=\frac{2}{3}R^2H. \tag{8}$$

Example 5. Calculate the volume enclosed by $2z=x^2+y^2$ and $x^2+y^2+z^2=3$.

Solution. We first understand the two surfaces. Clearly $x^2+y^2+z^2=3$ is a sphere centered at the origin with radius $\sqrt{3}$. On the other hand, we notice that if (x,y,z) belongs to the surface $2z=x^2+y^2$, so does (x',y',z) for every x', y' satisfying $(x')^2+(y')^2=x^2+y^2$. Therefore the surface $2z=x^2+y^2$ is obtained by rotating the graph of the parabola $z=\frac{x^2}{2}$ around the z axis.

Thus we see that the region is bounded below by the bowl shaped $2z=x^2+y^2$ and above by the sphere $x^2+y^2+z^2=3$. The two surfaces meet at points satisfying both equations and thus satisfy $3-z^2=2z \Longrightarrow z=-3,1$. The negative value does not make sense so $z=1$.

Therefore the region is given by

$$(x,y,z):\begin{cases} x^2+y^2\leqslant 2z & 0\leqslant z\leqslant 1\\ x^2+y^2\leqslant 3-z^2 & 1\leqslant z\leqslant\sqrt{3}\end{cases}. \tag{9}$$

Thus if we cut the region with plane passing $(0,0,z)$, the area is given by

$$A(z)=\begin{cases} 2\pi z & 0\leqslant z\leqslant 1\\ \pi(3-z^2) & 1\leqslant z\leqslant\sqrt{3}\end{cases}. \tag{10}$$

The volume is then given by

$$V=\int_0^{\sqrt{3}}A(z)\,\mathrm{d}z=\int_0^1 2\pi z\,\mathrm{d}z+\int_1^{\sqrt{3}}\pi(3-z^2)\,\mathrm{d}z=\frac{6\sqrt{3}-5}{3}\pi. \tag{11}$$