Math 118 Winter 2015 Lecture 38 (Mar. 20, 2015)

- Convex and Concave Functions

Definition 1. (Convex functions) *A function $f: [a, b] \mapsto \mathbb{R}$ is convex if and only if*

$$\forall x, y \in [a, b], \quad \forall \lambda \in [0, 1], \qquad f(\lambda\, x + (1 - \lambda)\, y) \leqslant \lambda\, f(x) + (1 - \lambda)\, f(y). \tag{1}$$

Definition 2. (Concave functions) *A function $f: [a, b] \mapsto \mathbb{R}$ is concave if and only if*

$$\forall x, y \in [a, b], \quad \forall \lambda \in [0, 1], \qquad f(\lambda\, x + (1 - \lambda)\, y) \geqslant \lambda\, f(x) + (1 - \lambda)\, f(y). \tag{2}$$

Exercise 1. What kind of function is both convex and concave over an interval $[a, b]$?

Exercise 2. What kind of function satisfies $\forall x,\ y \in [a,\ b], \quad \forall \lambda \in \mathbb{R}, \qquad f(\lambda\ x + (1 - \lambda)\ y) \leqslant \lambda\, f(x) + (1 - \lambda)\, f(y)$?

Proposition 3. *$f: [a, b] \mapsto \mathbb{R}$ is convex if and only if $g(x) := -f(x)$ is concave.*

Proof. We prove "only if" and leave "if" as exercise.

Let $f: [a, b] \mapsto \mathbb{R}$ be convex. Let $x, y \in [a, b], \lambda \in [0, 1]$ be arbitrary. Then by definition of g and convexity of f we have

$$\begin{aligned} g(\lambda\, x + (1 - \lambda)\, y) &= -f(\lambda\, x + (1 - \lambda)\, y) \\ &\geqslant -[\lambda\, f(x) + (1 - \lambda)\, f(y)] \\ &= \lambda\, g(x) + (1 - \lambda)\, g(y). \end{aligned} \tag{3}$$

Thus ends the proof. □

Exercise 3. Prove the "if" part.

- The role of convexity in optimization.

Theorem 4. *Let $f(x)$ be convex on $[a, b]$. Then any local minimizer of the problem*

$$\min f(x) \qquad \textit{subject to } a \leqslant x \leqslant b \tag{4}$$

is also global.

Proof. Let x_0 be a local minimizer of the problem. Assume that it is not global. Then there is $x_1 \in [a, b]$ such that $f(x_1) < f(x_0)$. Wlog assume $x_1 > x_0$. We also assume that $x_0 \in (a, b)$ and leave the cases $x_0 = a, x_0 = b$ as exercises.

Let $\delta > 0$ be arbitrary. There is $\delta_1 \in (0, \delta)$ such that $x_0 + \delta_1 \in (x_0, x_1)$. We find $\lambda \in [0, 1]$ such that $x_0 + \delta_1 = \lambda\, x_0 + (1 - \lambda)\, x_1$. Moving x_0 to the right hand side we easily obtain $\delta_1 = (1 - \lambda)\, (x_1 - x_0)$ and therefore

$$x_0 + \delta_1 = \lambda\, x_0 + (1 - \lambda)\, x_1 \tag{5}$$

for $\lambda = \frac{x_1 - x_0 - \delta_1}{x_1 - x_0} \in (0, 1)$. Now by convexity of f we have

$$f(x_0 + \delta_1) \leqslant \lambda\, f(x_0) + (1 - \lambda)\, f(x_1) < \lambda\, f(x_0) + (1 - \lambda)\, f(x_0) = f(x_0), \tag{6}$$

a contradiction to the fact that x_0 is a local minimizer. □

Exercise 4. Write down the detailed proof for the case $x_1 < x_0$.

- Properties of convex functions.

  1. f is convex on $[a, b] \Longleftrightarrow \forall x_1, \ldots, x_n \in [a, b], \forall \lambda_1, \ldots, \lambda_n \geqslant 0, \lambda_1 + \cdots + \lambda_n = 1$,

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \leqslant \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n). \tag{7}$$

Proof. We prove $\Longrightarrow$ and leave $\Longleftarrow$, which is trivial, as an exercise.

We prove by induction. The base case $n = 2$ is exactly the definition of convexity. Now assume that (7) holds for $n - 1$.

Let $x_1, \ldots, x_n \in [a, b]$ be arbitrary, $\lambda_1, \ldots, \lambda_n \geqslant 0$ be arbitrary but satisfying $\lambda_1 + \cdots + \lambda_n = 1$.

If one of $\lambda_i = 0$ then the situation reduces to the case $n - 1$. In the following we assume $\lambda_i > 0$ for all i. Now by definition of convexity and the induction hypothesis we have

$$\begin{aligned} f(\lambda_1 x_1 + \cdots + \lambda_n x_n) &= f\left(\lambda_1 x_1 + (1 - \lambda_1) \frac{\lambda_2 x_2 + \cdots + \lambda_n x_n}{1 - \lambda_1}\right) \\ &\leqslant \lambda_1 f(x_1) + (1 - \lambda_1) f\left(\frac{\lambda_2}{1 - \lambda_1} x_2 + \cdots + \frac{\lambda_n}{1 - \lambda_1} x_n\right) \\ &\leqslant \lambda_1 f(x_1) + (1 - \lambda_1) \left[\frac{\lambda_2}{1 - \lambda_1} f(x_2) + \cdots + \frac{\lambda_n}{1 - \lambda_1} f(x_n)\right] \\ &= \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n). \end{aligned} \tag{8}$$

There is one small gap in this proof which is left as exercise. □

Exercise 5. Why is $\frac{\lambda_2 x_2 + \cdots + \lambda_n x_n}{1 - \lambda_1} \in [a, b]$?

  2. f is convex on $[a, b] \Longleftrightarrow \forall a \leqslant x < y < z \leqslant b$, $\frac{f(z) - f(y)}{z - y} \geqslant \frac{f(y) - f(x)}{y - x} \Longleftrightarrow \forall a \leqslant x < y < z \leqslant b$, $\frac{f(z) - f(x)}{z - x} \geqslant \frac{f(y) - f(x)}{y - x}$.

Proof. We prove the second $\Longrightarrow$ and leave others as exercises.

Let $a \leqslant x < y < z \leqslant b$. We first find $\lambda \in (0, 1)$ such that $y = \lambda x + (1 - \lambda) z$. Subtracting x from both sides we have $y - x = (1 - \lambda)(z - x) \Longrightarrow \lambda = \frac{z - y}{z - x}$. By convexity of f we have

$$f(y) \leqslant \frac{z - y}{z - x} f(x) + \frac{y - x}{z - x} f(z) \tag{9}$$

which simplifies to $\frac{f(z) - f(x)}{z - x} \geqslant \frac{f(y) - f(x)}{y - x}$. □

Exercise 6. Prove the remaining relations.

  3. f is convex on $[a, b]$, then for every $x_0 \in (a, b)$, $\lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0}$ and $\lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0}$ exist and are finite. In particular, $f(x)$ is continuous on (a, b).

Proof. Let $x_0 \in (a, b)$ be arbitrary. Consider the function $F: (x_0, b) \mapsto \mathbb{R}$

$$F(x) := \frac{f(x) - f(x_0)}{x - x_0}. \tag{10}$$

From the above property we see that

  i. F is increasing;

ii. $F(x) \geqslant \frac{f(a) - f(x_0)}{a - x_0}$ for all $x \in (x_0, b)$.

Therefore $\lim_{x \to x_0 +} F(x)$ exists and is finite. The proof for the left limit is similar. □

Exercise 7. Prove that f is continuous on (a, b).

Remark 5. Note that f does not need to be continuous on $[a, b]$.

Exercise 8. Prove that $f(x) = \begin{cases} x^2 & x \in (-1, 1) \\ 2 & x = \pm 1 \end{cases}$ is convex.

4. The following are simple consequences of Property 2:

THEOREM 6. *Let f be differentiable on (a, b). Then f is convex on (a, b) if and only if $f'(x)$ is increasing on (a, b).*

Exercise 9. Prove Theorem 6.

THEOREM 7. *Let f be twice ifferentiable on (a, b). Then f is convex on (a, b) if and only if $f''(x) \geqslant 0$ on (a, b).*

Exercise 10. Prove Theorem 7.

Example 8. Applying Theorem 7 it is trivial to prove that $f(x) = -\ln x$ is convex on $(0, \infty)$. Now taking arbitrary $r_1, \ldots, r_n > 0$ and setting $\lambda_i = \frac{1}{n}$ for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} -\ln\left(\frac{r_1 + \cdots + r_n}{n}\right) &\leqslant -\frac{1}{n} [\ln(r_1) + \cdots + \ln(r_n)] \\ &= -\ln\left[(r_1 \cdots r_n)^{1/n}\right]. \end{aligned} \tag{11}$$

Exercise 11. Prove that

$$(r_1 \cdots r_n)^{1/n} \leqslant \frac{r_1 + \cdots + r_n}{n}. \tag{12}$$

Exercise 12. Let $a, b > 0$ be arbitrary. Let $p > 1$ and $q := \frac{p}{p-1}$. Prove Young's inequality:

$$a\, b \leqslant \frac{a^p}{p} + \frac{b^q}{q}. \tag{13}$$

Then prove Hölder's inequality: $\forall x_1, \ldots, x_n, y_1, \ldots, y_n > 0,\ p > 1, q = \frac{p}{p-1}$,

$$\sum_{k=1}^n x_k\, y_k \leqslant \left(\sum_{k=1}^n x_k^p\right)^{1/p} \left(\sum_{k=1}^n y_k^q\right)^{1/q}. \tag{14}$$

(Hint:[1])

1. First show that wlog we can assume $\sum_{k=1}^n x_k^p = \sum_{k=1}^n y_k^q = 1$.