Math 118 Winter 2015 Homework 8 Solutions

Due Thursday Mar. 26 3pm in Assignment Box

Question 1. (5 pts) *Solve the following optimization problems (min/max means you need to solve both the minimization and maximization problems)*

a) $\min/\max f(x) = \frac{x^3}{3} - 2x^2 + 3x + 1$ *subject to* $-1 \leqslant x \leqslant 5$*;*

b) $\min/\max f(x) = -3x^4 + 6x^2 - 1$ *subject to* $-2 \leqslant x \leqslant 2$*.*

Solution.

a) $x_{\max} = 5, x_{\min} = -1$;

b) $x_{\max} = \pm 1, x_{\min} = \pm 2$.

Question 2. (5 pts) *Prove: Among all rectangles inside a fixed circle, the inscribed square has the maximum area and perimeter.*

Proof. It is clear that we can consider only inscribed rectangles. Wlog assume the radius of the circle is 1.

- Area.

  Let the sides of the rectangle by a, b, then we are solving

$$\max a\, b \qquad \text{subject to } a^2 + b^2 = 4,\ a \geqslant 0,\ b \geqslant 0. \tag{1}$$

  As the constraint is equivalent to $b = \sqrt{4 - a^2}$, $0 \leqslant a \leqslant 2$ the problem is equivalent to

$$\max f(a) := a\sqrt{4 - a^2} \qquad \text{subject to } 0 \leqslant a \leqslant 2. \tag{2}$$

  We calculate

$$f'(a) = \frac{4 - 2a^2}{\sqrt{4 - a^2}} \tag{3}$$

  so $f'(a) = 0 \Longrightarrow a_0 = \sqrt{2} \in [0, 2]$. We compare

$$f(0) = 0, \qquad f(2) = 0, \qquad f\left(\sqrt{2}\right) = 2. \tag{4}$$

  Therefore the global maximum is reached at $a = b = \sqrt{2}$ which is the inscribed square.

- Perimeter. Similarly we solve

$$\max f(a) := a + \sqrt{4 - a^2} \qquad \text{subject to } 0 \leqslant a \leqslant 2. \tag{5}$$

  This time we have

$$f'(a) = 1 - \frac{a}{\sqrt{4 - a^2}}. \tag{6}$$

  Solving $f'(a) = 0$ gives $a_0 = \sqrt{2}$. We compare

$$f(0) = 2, \qquad f(2) = 2, \qquad f\left(\sqrt{2}\right) = 2\sqrt{2}. \tag{7}$$

  Thus $a = \sqrt{2}$ is the global maximizer and the conclusion follows. □

QUESTION 3. (5 PTS) *Let $f(x)$ be infinitely differentiable on $\mathbb{R}$. Consider*

$$\min f(x) \qquad \textit{subject to } -\infty < x < \infty. \tag{8}$$

a) (2 PTS) *Assume $f'(0) = f''(0) = f'''(0) = 0$ and $f^{(4)}(0) > 0$. What can we conclude about 0? A) local minimizer; B) local maximizer; C) neither; D) cannot decide.*

b) (3 PTS) *Assume $f'(0) = f''(0) = 0$ and $f'''(0) < 0$. What can we conclude about 0? A) local minimizer; B) local maximizer; C) neither; D) cannot decide.*

Justify your answers (using only results from our lecture notes).

Solution.

a) 0 is a local minimizer. As $\lim_{x\to 0} \frac{f'''(x) - f'''(0)}{x - 0} = f^{(4)}(0) > 0$ there is $\delta > 0$ such that

$$x \in (-\delta, \delta) - \{0\} \Longrightarrow \frac{f'''(x)}{x} > 0 \tag{9}$$

which gives

$$f'''(x) < 0 \text{ for } x \in (-\delta, 0); \qquad f'''(x) > 0 \text{ for } x \in (0, \delta). \tag{10}$$

Now let $x \in (-\delta, 0)$ be arbitrary. By Taylor's theorem we have

$$f(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2}\, x^2 + \frac{f'''(c)}{6}\, x^3 = f(0) + \frac{f'''(c)}{6}\, x^3 \tag{11}$$

for some $c \in (x, 0) \subset (-\delta, 0)$. Therefore $f'''(c) < 0$ and $\frac{f'''(c)}{6}\, x^3 > 0$. Thus we have $f(x) > f(0)$ for $x \in (-\delta, 0)$. Similarly we prove $f(x) > f(0)$ for $x \in (0, \delta)$.

b) 0 is neither. We prove that it is not a local minimizer. The proof for it being not a local maximizer is almost identical.

Let $\delta > 0$ be arbitrary. As $\lim_{x\to 0} \frac{f''(x) - f''(0)}{x - 0} = f'''(0) < 0$ there is $\delta_1 > 0$ such that

$$\forall x \in (-\delta_1, \delta_1) - \{0\}, \qquad \frac{f''(x)}{x} < 0 \tag{12}$$

which gives

$$f''(x) > 0 \text{ for } x \in (-\delta_1, 0); \qquad f''(x) < 0 \text{ for } x \in (0, \delta_1). \tag{13}$$

Now take $x_1 \in (0, \delta_1) \cap (0, \delta)$. By Taylor's theorem we have

$$f(x_1) = f(0) + f'(0)\, x_1 + \frac{f''(c)}{2}\, x_1^2 = f(0) + \frac{f''(c)}{2}\, x_1^2 \tag{14}$$

for some $c \in (0, x_1) \subset (0, \delta)$. As $c \in (0, x_1) \subset (0, \delta_1)$, $f''(c) < 0 \Longrightarrow f(x_1) < f(0)$. Therefore 0 is not a local minimizer.

QUESTION 4. (5 PTS) *Let $f(x)$ be continuous, strictly increasing on $[0, a]$ for some $a > 0$ with $f(0) = 0$. Let $g(x)$ be its inverse function. Prove the following inequality:*

$$\forall x \in [0, a], y \in [0, f(a)], \qquad x\, y \leqslant \int_0^x f(t)\, \mathrm{d}t + \int_0^y g(u)\, \mathrm{d}u. \tag{15}$$

(Hint: Consider $\max F(x) := x\, y - \int_0^x f(t)\, \mathrm{d}t$*.)*

Proof. Let $y \in [0, f(a)]$ be fixed and set $F(x) := x\, y - \int_0^x f(t)\, \mathrm{d}t$.

Solving $F'(x)=0$ we have $x_0=g(y)$. As $F'(x)=y-f(x)$ and $f(x)$ is strictly increasing, $F'(x)>0$ when $x<x_0$ and $F'(x)<0$ when $x>x_0$. Consequently

$$\forall x\in[0,a], \qquad F(x)\leqslant F(x_0)=y\,g(y)-\int_0^{g(y)} f(t)\,\mathrm{d}t. \tag{16}$$

Now we calculate $F(x_0)$. Making the change of variable $t=g(u)$ and then integrate by parts we have

$$\begin{aligned} F(x_0) &= y\,g(y)-\int_0^{g(y)} f(t)\,\mathrm{d}t \\ &= y\,g(y)-\int_0^{y} f(g(u))\,g'(u)\,\mathrm{d}i \\ &= y\,g(y)-\int_0^{y} u\,\mathrm{d}g(u) \\ &= y\,g(y)-y\,g(y)+\int_0^{y} g(u)\,\mathrm{d}u=\int_0^{y} g(u)\,\mathrm{d}u \end{aligned} \tag{17}$$

Thus ends the proof. □