Math 118 Winter 2015 Lecture 37 (Mar. 19, 2015)

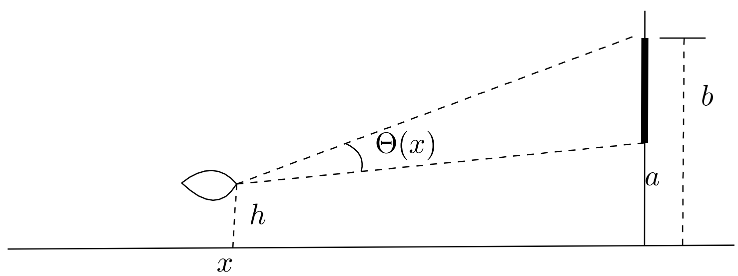
- Regiomontanus Problem.

  The following problem was posed by Johannes Mueller[1] (1436–1476, aka Regiomontanus).

  > A painting hangs from a wall. Given the heights of the top and bottom of the painting above the viewer's eye level, how far from the wall should the viewer stand in order to maximize the angle subtended by the painting and whose vertex is at the viewer's eye?[2]

  Note that this problem only makes sense when the lower edge of the painting is higher than the eye level of the viewer or the upper edge is lower than the eye level. Otherwise the angle is clearly maximized when the viewer is at distance 0 from the wall which is, unfortunately, nonsensical. In the following we assume the lower edge is higher than eye level.

  Denote by h the height of the eye and $b > a > h$ the height of the lower, upper edges of the painting. Let $x \in [0, \infty)$ be the distance between the viewer and the wall.

  So the problem is

  $$\max \Theta(x) \qquad \text{subject to } 0 \leqslant x < \infty. \tag{1}$$

  We easily obtain

  $$\Theta(x) = \arctan \frac{b-h}{x} - \arctan \frac{a-h}{x}. \tag{2}$$

  We calculate

  $$\Theta'(x) = \frac{a-h}{x^2 + (a-h)^2} - \frac{b-h}{x^2 + (b-h)^2}. \tag{3}$$

  Setting this to 0 we reach $x = \pm\sqrt{(a-h)\,(b-h)}$. As $-\sqrt{(a-h)\,(b-h)}$ does not satisfy the constraint, the only candidate for the interior local maximizer is $\sqrt{(a-h)\,(b-h)}$.

  Now notice that $\Theta(0) = \Theta(\infty) = 0$ while $\Theta\left(\sqrt{(a-h)\,(b-h)}\right) > 0$.

  Exercise 1. Prove from the above information that $\sqrt{(a-h)\,(b-h)}$ is the global maximizer.

  Thus the solution to the problem is $x = \sqrt{(a-h)\,(b-h)}$.

  Remark 1. A more realistic setting is to remember that a painting has not only height but also width and therefore it makes more sense to maximize the "Euler angle" of the painting. This becomes a multi-variable calculus problem. See §3.5 of *When Least is Best* by Paul J. Nahin for some discussion on this problem.

- Kepler's Wine Barrel. [3]

1. http://en.wikipedia.org/wiki/Regiomontanus
2. http://en.wikipedia.org/wiki/Regiomontanus'_angle_maximization_problem.
3. http://www.maa.org/publications/periodicals/convergence/kepler-the-volume-of-a-wine-barrel.

"Shortly after his second marriage in 1613, while setting up a new household, he learned how wine merchants determined the 'volume' of wine barrels. They simply stuck a rod in through a hole at the edge of the top lid and measured the length of the barrel diagonal from top to bottom, without regard to the actual shape of the barrel. This made no sense to a man with Kepler's mathematical ability, of course, and he began to think upon the question of just how one would compute the volumes of various barrel shapes." [4]

We are solving the following problem.

$$\max V = \pi r^2 h \qquad \text{subject to } (2r)^2 + h^2 = l^2. \tag{4}$$

This is equivalent to

$$\max V(h) := \pi \frac{l^2 - h^2}{4} h \qquad \text{subject to } 0 \leqslant h \leqslant l. \tag{5}$$

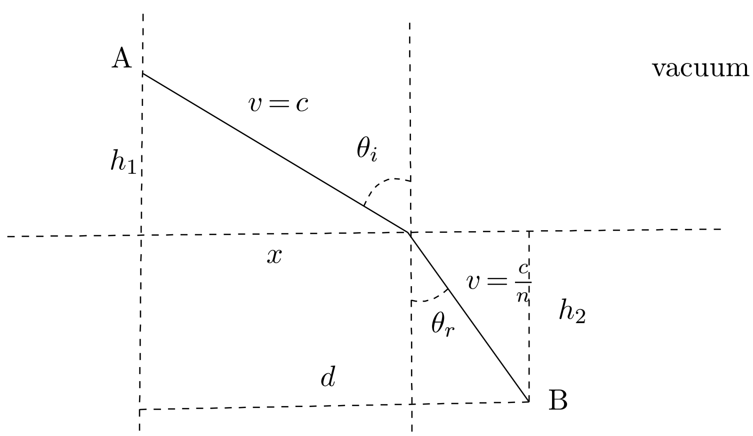
Taking derivative and setting to 0 we have

$$V'(h) = 0 \Longrightarrow h = \frac{l}{\sqrt{3}}. \tag{6}$$

As $V(0) = V(l) = 0$, similar to the previous Regiomontanus problem, from this we can show that $h = \frac{l}{\sqrt{3}}$ is the unique global maximizer.

- Snell's Law.

  Consider light traveling from a point A in vacuum to a point B in some substance where the speed of light is c/n.



  We solve the problem

$$\min T(x) := \frac{\sqrt{h_1^2 + x^2}}{c} + \frac{\sqrt{h_2^2 + (d-x)^2}}{c/n} \text{ subject to } 0 \leqslant x \leqslant d. \tag{7}$$

We have

$$T'(x) = \frac{x}{c\sqrt{h_1^2 + x^2}} - \frac{n(d-x)}{c\sqrt{h_2^2 + (d-x)^2}}. \tag{8}$$

Setting $T'(x) = 0$ we obtain, for the solution x_0:

$$\frac{x_0}{\sqrt{h_1^2 + x_0^2}} = n \frac{d - x_0}{\sqrt{h_2^2 + (d - x_0)^2}} \tag{9}$$

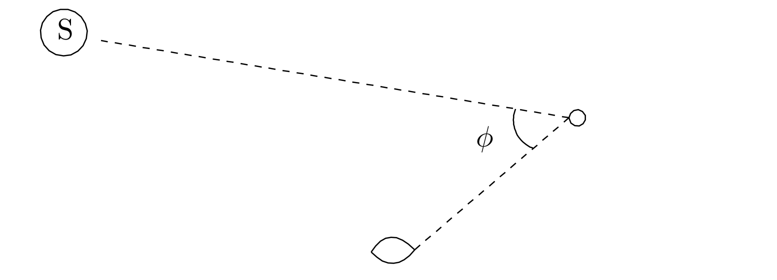
4. Paul J. Nahin, *When Least is Best*, p.108.

which is exactly Snell's Law:

$$\frac{\sin\theta_i}{\sin\theta_r} = n. \tag{10}$$

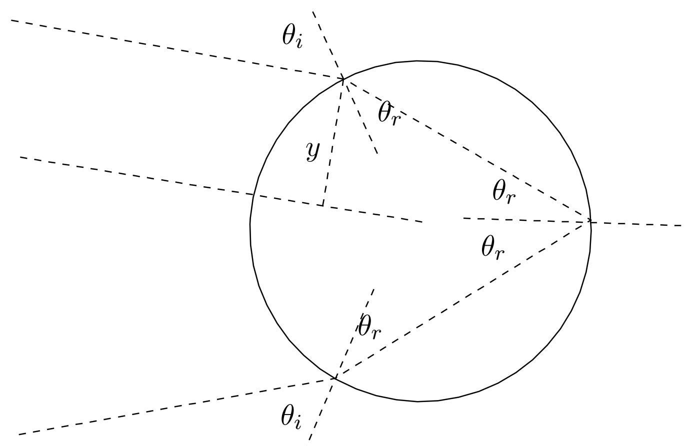
Problem 1. We have only shown $T'(x_0) = 0$. Prove that this x_0 is indeed the global minimizer for the problem.

- Formation of Rainbows.



$\phi_1 \approx 42^o$ for primary rainbow, $\phi_2 \approx 52^o$ for secondary rainbow, $\phi_3 \approx 318^o$ for tertiary rainbow. Interestingly the tertiary rainbow is a halo around the sun.

Primary rainbow is formed when there is only one reflection inside the raindrop:



We see that the angle between the incoming and the out-going rays, $\phi = 4\,\theta_r - 2\,\theta_i$. By Snell's Law we have

$$\phi = 4\arcsin\left(\frac{1}{n}\sin\theta_i\right) - 2\,\theta_i. \tag{11}$$

Now it is reasonable to assume that the energy of the incoming sun light rays is uniformly distributed in y. Noticing that $\sin\theta_i = \frac{y}{R}$ where R is the radius of the raindrop, we see that

$$\phi(y) = 4\arcsin\left(\frac{y}{n\,R}\right) - 2\arcsin\left(\frac{y}{R}\right), \qquad 0 \leqslant y \leqslant R. \tag{12}$$

Taking derivative and setting to 0 we have, at the local maximizer,

$$\frac{2}{n}\sqrt{1-\left(\frac{y}{R}\right)^2} = \sqrt{1-\left(\frac{y}{n\,R}\right)^2} \Longrightarrow \frac{4}{n^2}\left[1-\left(\frac{y}{R}\right)^2\right] = 1-\left(\frac{y}{n\,R}\right)^2. \tag{13}$$

But this means $\frac{4}{n^2}\cos^2\theta_i = \cos^2\theta_r$. Together with $\sin\theta_i = n\sin\theta_r$ we can solve

$$\theta_i = \arccos\left(\sqrt{\frac{n^2-1}{3}}\right). \tag{14}$$

For water $n \approx \frac{4}{3} \Longrightarrow \phi \approx 42^o$. It turns out that $\phi \in [40^o, 42^o]$ (roughly) when $\frac{y}{R} \in [0.75, 0.95]$. Thus about 20% of the incoming energy comes out of the raindrop around the angle 42^o and this is why we can see a rainbow there.

Remark 2. If we take into account that n differs with the wavelength, and thus with the color of the light, we can calculate $\phi_{\text{red}} \approx 42.37^o$ while $\phi_{\text{violet}} \approx 40.5^o$.

Remark 3. See §5.8 of *When Least is Best* by Paul J. Nahin (and the references therein) for more discussions on the formation of rainbows.