Math 118 Winter 2015 Homework 7 Solutions

Due Thursday Mar. 12 3pm in Assignment Box

Question 1. (5 pts) *Let $a < c < b$. Assume $f_n \longrightarrow f$ uniformly on $[a, c]$ and $[c, b]$. Prove that $f_n \rightarrow f$ uniformly on $[a, b]$.*

Proof. Let $\varepsilon > 0$ be arbitrary. As $f_n \longrightarrow f$ uniformly on $[a, c]$ there is $N_1 \in \mathbb{N}$ such that

$$\forall n > N_1, \forall x \in [a, c], \qquad |f_n(x) - f(x)| < \varepsilon; \tag{1}$$

As $f_n \longrightarrow f$ uniformly on $[c, b]$, there is $N_2 \in \mathbb{N}$ such that

$$\forall n > N_2, \forall x \in [c, b], \qquad |f_n(x) - f(x)| < \varepsilon. \tag{2}$$

Now set $N = \max\{N_1, N_2\}$. Then for every $n > N$, we have $|f_n(x) - f(x)| < \varepsilon$ for every $x \in [a, c]$ and every $x \in [c, b]$, that is for every $x \in [a, b]$. The proof ends. $\square$

Question 2. (5 pts) *Let $f(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2}$. Prove that f is continuous on $[0, \infty)$ and furthermore $\int_0^1 f(x)\,\mathrm{d}x = 1$.*

Proof. It is clear that $\forall x \in [0, \infty)$, $\left|\frac{1}{(x+n)^2}\right| \leqslant \frac{1}{n^2}$. As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the convergence of $\sum_{n=1}^{\infty} \frac{1}{(x+n)^2}$ is uniform. The continuity of f now follows from the continuity of each $\frac{1}{(x+n)^2}$.

Since each $\frac{1}{(x+n)^2}$ is integrable on $[0, 1]$, we have

$$\begin{aligned} \int_0^1 f(x)\,\mathrm{d}x &= \sum_{n=1}^{\infty} \int_0^1 \frac{1}{(x+n)^2}\,\mathrm{d}x \\ &\overset{u=x+n}{=\!=\!=} \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{u^2}\,\mathrm{d}u \\ &= \int_1^{\infty} \frac{1}{u^2}\,\mathrm{d}u \\ &= 1. \end{aligned} \tag{3}$$

Thus ends the proof. $\square$

Question 3. (5 pts) *Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two power series. Assume that there is $r > 0$ such that*

$$\forall |x| < r, \qquad \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n. \tag{4}$$

Prove $\forall n \in \mathbb{N} \cup \{0\}$, $a_n = b_n$.

Proof. Let $f(x) := \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ on $(-r, r)$. As both are power series, the radii of convergence are both $\geqslant r$. Consequently on $(-r, r)$ we have

$$f'(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1} = \sum_{n=1}^{\infty} n\, b_n x^{n-1}, \tag{5}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)\, b_n x^{n-2}, \tag{6}$$

$$\vdots \quad \vdots \quad \vdots$$

Setting $x=0$ we have $a_1 = f'(0) = b_1$, $a_2 = f''(0) = b_2, \ldots$. □

QUESTION 4. (5 PTS) *Let* $f(x) := \sum_{n=1}^{\infty} \frac{\sin(n\,x)}{n^2}$. *Is* $f(x)$ *improperly integrable on* $(0,\infty)$*? Justify.*

Solution. No. First notice that $f(x)$ is periodic, that is $f(x+2\,\pi) = f(x)$ for all $x \in \mathbb{R}$.

Next we calculate

$$f\left(\frac{\pi}{2}\right) = \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\,\pi}{2}\right)}{n^2} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots \tag{7}$$

Let S_n be the partial sum of the first n terms of this series. Then clearly $S_{2n} \geqslant 1 - \frac{1}{3^2} = \frac{8}{9}$ for all $n \in \mathbb{N}$. Consequently $f\left(\frac{\pi}{2}\right) \geqslant \frac{8}{9}$.

As $\left|\frac{\sin(n\,x)}{n^2}\right| \leqslant \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the convergence of $\sum_{n=1}^{\infty} \frac{\sin(n\,x)}{n^2}$ is uniform and $f(x)$ is thus continuous. Therefore there is $\delta \in \left(0, \frac{\pi}{2}\right)$ such that $f(x) \geqslant \frac{1}{2}$ for all $x \in \left(\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta\right)$.

Now let $F(d) := \int_0^d f(x)\,\mathrm{d}x$. We will prove that $F(d)$ is not Cauchy which implies that $f(x)$ is not improperly integrable on $(0,\infty)$.

To show this, let $d_0 > 0$ be arbitrary. There is $n \in \mathbb{N}$ such that $2\,n\,\pi > d_0$. Now set $d_1 = 2\,n\,\pi + \frac{\pi}{2} - \delta$, $d_2 = 2\,n\,\pi + \frac{\pi}{2} + \delta$. We have $d_2 > d_1 > d$ and

$$|F(d_2) - F(d_1)| = \left|\int_{2n\pi + \frac{\pi}{2} - \delta}^{2n\pi + \frac{\pi}{2} + \delta} f(x)\,\mathrm{d}x\right| = \left|\int_{\frac{\pi}{2} - \delta}^{\frac{\pi}{2} + \delta} f(x)\,\mathrm{d}x\right| = \int_{\frac{\pi}{2} - \delta}^{\frac{\pi}{2} + \delta} f(x)\,\mathrm{d}x \geqslant \delta. \tag{8}$$

Thus $F(d)$ is not Cauchy and the proof ends.