MATH 118 WINTER 2015 LECTURE 29 (MAR. 4, 2015)

- Recall the theory of power series.

THEOREM 1. *Let* $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ *be a power series with radius of convergence* R. *Define* $f(x) := \sum_{n=0}^{\infty} a_n (x-x_0)^n$. *Then*

a) $f(x)$ *is continuous on* $(x_0 - R, x_0 + R)$.

b) $f(x)$ *is differentiable on* $(x_0 - R, x_0 + R)$, *and* $f'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-x_0)^n$ *on* $(x_0 - R, x_0 + R)$.

c) $f(x)$ *is infinitely differentiable on* $(x_0 - R, x_0 + R)$.

d) *Let* $[a, b] \subset (x_0 - R, x_0 + R)$ *be arbitrary. Then* $f(x)$ *is integrable on* $[a, b]$ *and furthermore*

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} a_n \int_a^b (x-x_0)^n\,\mathrm{d}x. \tag{1}$$

- Examples for power series.

Example 2. Calculate $\sum_{n=1}^{\infty} n\,x^n$.

Solution. It is easy to see that the radius of convergence is $R=1$, and the series diverges at $x=-1, 1$. Thus in the following we only consider $x \in (-1, 1)$.

Recall that for such x,

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x} \xrightarrow{\text{Differentiate}} \sum_{n=1}^{\infty} n\,x^{n-1} = \frac{1}{(1-x)^2}. \tag{2}$$

Therefore

$$\sum_{n=1}^{\infty} n\,x^n = \frac{x}{(1-x)^2} \tag{3}$$

for all $x \in (-1, 1)$.

Exercise 1. Prove

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \ln(1+x); \qquad \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}. \tag{4}$$

for $x \in (-1, 1)$.

Example 3. Let $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$. We will find a formula for the general F_n.

Consider the power series $\sum_{n=0}^{\infty} F_n\,x^n$.

Exercise 2. Prove that $0 \leqslant F_n \leqslant 2^{n-1}$.

We see that the radius of convergence $R \geqslant \frac{1}{2}$. Thus we consider $f(x) := \sum_{n=0}^{\infty} F_n x^n$ which is defined on $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Now we have

$$
\begin{aligned}
(1 - x - x^2) f(x) &= \sum_{n=0}^{\infty} F_n x^n - x \sum_{n=0}^{\infty} F_n x^n - x^2 \sum_{n=0}^{\infty} F_n x^n \\
&= \sum_{n=0}^{\infty} F_n x^n - \sum_{n=1}^{\infty} F_{n-1} x^n - \sum_{n=2}^{\infty} F_{n-2} x^n \\
&= F_0 + F_1 x + \sum_{n=2}^{\infty} F_n x^n - F_0 x - \sum_{n=2}^{\infty} F_{n-1} x^n - \sum_{n=2}^{\infty} F_{n-2} x^n \\
&= 1 + \sum_{n=2}^{\infty} (F_n - F_{n-1} - F_{n-2}) x^n \\
&= 1 \Longrightarrow f(x) = \frac{1}{1 - x - x^2}. \qquad (5)
\end{aligned}
$$

As $1 - x - x^2 = 0 \Longrightarrow x_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$, we can write

$$\frac{1}{1 - x - x^2} = \frac{A}{x + \frac{\sqrt{5} + 1}{2}} + \frac{B}{x - \frac{\sqrt{5} - 1}{2}} \tag{6}$$

and solve

$$A = \frac{1}{\sqrt{5}}, \qquad B = -\frac{1}{\sqrt{5}}. \tag{7}$$

Now we calculate

$$\frac{1}{\sqrt{5}} \frac{1}{x + \frac{\sqrt{5} + 1}{2}} = \frac{1}{\sqrt{5}} \frac{2}{\sqrt{5} + 1} \sum_{n=0}^{\infty} \left(-\frac{2}{\sqrt{5} + 1}\right)^n x^n \tag{8}$$

$$\frac{1}{\sqrt{5}} \frac{1}{x - \frac{\sqrt{5} - 1}{2}} = -\frac{1}{\sqrt{5}} \frac{2}{\sqrt{5} - 1} \sum_{n=0}^{\infty} \left(\frac{2}{\sqrt{5} - 1}\right)^n x^n. \tag{9}$$

Thus we have

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{\sqrt{5} + 1}{2}\right)^{n+1} - (-1)^{n+1} \left(\frac{\sqrt{5} - 1}{2}\right)^{n+1}\right]. \tag{10}$$

Example 4. Define

$$E(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{11}$$

Then $E(x + y) = E(x) E(y)$.

Exercise 3. Prove that $E(x)$ is defined for all $x \in \mathbb{R}$ and $E'(x) = E(x)$.

Proof. First we notice that

$$E(x) \geqslant 1 - |x| - |x|^2 - \cdots = 1 - \frac{|x|}{1 - |x|} > 0 \tag{12}$$

for all $|x| < \frac{1}{2}$. Now fix an arbitrary $y \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and consider $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Let

$$f(x) := \frac{E(x + y)}{E(x) E(y)}. \tag{13}$$

Then we have $f(0)=1$ and

$$f'(x)=\frac{E(x+y)\,E(x)\,E(y)-E(x)\,E(y)\,E(x+y)}{E(x)^2\,E(y)^2}=0 \tag{14}$$

therefore $E(x+y)=E(x)\,E(y)$ for all $x,y\in\left(-\frac{1}{2},\frac{1}{2}\right)$. This implies $E(x)>0$ for all $x\in(-1,1)$. Repeating the above argument we have $E(x+y)=E(x)\,E(y)$ for all $x,y\in(-1,1)$ and so on. □

Exercise 4. Define

$$S(x):=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2\,n+1)!}\,x^{2n+1},\qquad C(x):=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2\,n)!}\,x^{2n}. \tag{15}$$

Prove

a) $S(x)$ and $C(x)$ are both infinitely differentiable on $\mathbb{R}$;

b) $S'(x)=C(x), C'(x)=-S(x)$;

c) $S(x+y)=S(x)\,C(y)+C(x)\,S(y)$; $C(x+y)=C(x)\,C(y)-S(x)\,S(y)$;

d) $S^2(x)+C^2(x)=1$;

e) There is exactly one $x_0>0$ such that $C(x_0)=0$, $C(x)>0$ for $x\in[0,x_0)$; (Hint:[1])

f) $S(x_0)=1$;

g) $S(2\,x_0-x)=S(x)$, $C(2\,x_0-x)=-C(x)$;

h) $S(x)\neq 0$ on $(0,2\,x_0)$ and $(2\,x_0,4\,x_0)$. (Hint:[2])

i) $\forall x\in\mathbb{R}$, $S(x+4\,x_0)=S(x)$, $C(x+4\,x_0)=C(x)$, and $4\,x_0$ is the smallest positive number having this property.

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1. Apply MVT to $S(x)$ on $[0,2]$ to conclude there is $|C(\xi)|\leqslant\frac{1}{2}$. Then prove $C(2\,\xi)<0$. Then apply IVP to $C(\xi)$.
2. Assume the contrary. Apply MVT to conclude $C(\xi_1)=C(\xi_2)=0$. At least one ξ is different from x_0.