Math 118 Winter 2015 Homework 6 Solutions

Due Thursday Mar. 5 3pm in Assignment Box

Question 1. (5 pts) *Calculate the following.*

a) (2 pts) $\lim_{n\to\infty}\frac{1}{1+x^n}$ *on* $\{x|\,x\geqslant 0\}$;

b) (3 pts) $\lim_{n\to\infty}e^{-nx}\,(1+x^2)^n$ *on* $\{x|\,x\geqslant 0\}$.

Solution.

a) We know that

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & x\in[0,1) \\ 1 & x=1 \\ +\infty & x>1 \end{cases}. \tag{1}$$

Consequently

$$\lim_{n\to\infty}\frac{1}{1+x^n} = \begin{cases} 1 & x\in[0,1) \\ \frac{1}{2} & x=1 \\ 0 & x>1 \end{cases}. \tag{2}$$

b) When $x=0$ we have the limit to be 1. For an arbitrary $x>0$, we have $e^{-nx}\,(1+x^2)^n = [e^{-x}\,(1+x^2)]^n$. We see that it depends whether $e^{-x}\,(1+x^2)$ is greater than, equal to, or less than 1.

Let $u(x) := e^{-x}\,(1+x^2)$. We calculate

$$[e^{-x}\,(1+x^2)]' = [-(1+x^2)+2\,x]\,e^{-x} = -(x-1)^2\,e^{-x}\leqslant 0. \tag{3}$$

Therefore $u(x)$ is decreasing for all $x>0$. Furthermore for $x\in(0,1)$, we have $u'(x)<0$ which means $u(x)<u(0)=1$ for $x\in(0,1)$. Now for every $x\in[1,\infty)$ we have $u(x)<u(1/2)<1$.

Thus $u(x)\in(0,1)$ for all $x>0$ and consequently $\lim_{n\to\infty}u(x)^n=0$ for all $x>0$.

So

$$\lim_{n\to\infty} e^{-nx}\,(1+x^2)^n = \begin{cases} 1 & x=0 \\ 0 & x>0 \end{cases}. \tag{4}$$

Question 2. (5 pts) *Let* $f_n(x)$ *converge to* $f(x)$ *uniformly on* $[a,b]$. *Prove that* $\lim_{n\to\infty}f_n(x) = f(x)$ *on* $[a,b]$.

Proof. Let $x_0\in[a,b]$ be arbitrary. Let $\varepsilon>0$ be arbitrary. As $f_n(x)$ converge to $f(x)$ uniformly on $[a,b]$, there is $N_1\in\mathbb{N}$ such that for all $n>N_1$ and $x\in[a,b]$, $|f_n(x)-f(x)|<\varepsilon$. Now take $N=N_1$. We have for all $n>N$ and all $x\in[a,b]$, $|f_n(x)-f(x)|<\varepsilon$. In particular for all $n>N$, $|f_n\,(x_0)-f(x_0)|<\varepsilon$. Therefore $\lim_{n\to\infty}f_n(x_0)=f(x_0)$ and the conclusion follows. □

Question 3. (5 pts)

a) (2 pts) *Prove that* $\sum_{n=1}^{\infty} e^{-nx}\sin(n^2\,x)$ *converges on* $[0,\infty)$.

b) (3 pts) *Is the convergence uniform? Justify your claim.*

Solution.

a) When $x=0$ we have $e^{-nx}\sin(n^2 x)=0$ for all n so the the limit is 0. When $x>0$ we have

$$|e^{-nx}\sin(n^2 x)|\leqslant (e^{-x})^n. \tag{5}$$

As $x>0$, $e^{-x}\in(0,1)$ and therefore $\sum_{n=1}^{\infty}(e^{-x})^n$ converges. Consequently $\sum_{n=1}^{\infty} e^{-nx}\sin(n^2 x)$ converges. In summary, $\sum_{n=1}^{\infty} e^{-nx}\sin(n^2 x)$ converges on $[0,\infty)$.

b) We claim that the convergence is not uniform. To do this we show that $|e^{-nx}\sin(n^2 x)|$ does not converge to 0 uniformly. Let $N\in\mathbb{N}$ be arbitrary. Set $n=2N$ and $x=n^{-2}$. Then we have

$$|e^{-nx}\sin(n^2 x)-0| = e^{-\frac{1}{n}}\sin(1) > e^{-1}\sin(1) > 0. \tag{6}$$

Question 4. (5 pts) *Let* $f_n(x)$, $f(x)$: $[0,\infty)\mapsto\mathbb{R}$. *Further assume* $\lim_{x\to\infty} f_n(x)=0$ *for every* $n\in\mathbb{N}$ *and* $\lim_{n\to\infty} f_n(x)=f(x)$ *on* $[0,\infty)$.

a) (2 pts) *Does it follow that* $\lim_{x\to\infty} f(x)=0$*? Justify your claim.*

b) (3 pts) *If instead of convergence we assume* $f_n(x)$ *converges to* $f(x)$ *uniformly on* $[0,\infty)$*, does it follow that* $\lim_{x\to\infty} f(x)=0$*? Justify your claim.*

Solution.

a) No. Consider $f_n(x)=\begin{cases}1 & x\in[0,n]\\ 0 & x\in(n,\infty)\end{cases}$. Then we have $\lim_{x\to\infty} f_n(x)=0$ and $\lim_{n\to\infty} f_n(x)=f(x)=1$. But $\lim_{x\to\infty} f(x)=1\neq 0$.

b) Yes. Let $\varepsilon>0$ be arbitrary. As $f_n(x)$ converges to $f(x)$ uniformly on $[0,\infty)$, there is $N\in\mathbb{N}$ such that for all $n>N$ and all $x\in[0,\infty)$, $|f_n(x)-f(x)|<\frac{\varepsilon}{2}$. In particular, there is $n_0\in\mathbb{N}$ such that $|f_{n_0}(x)-f(x)|<\frac{\varepsilon}{2}$. As $\lim_{x\to\infty} f_{n_0}(x)=0$, there is $R_1>0$ such that for all $x>R_1$, $|f_{n_0}(x)|<\frac{\varepsilon}{2}$.

Now set $R=R_1$. For all $x>R$, we have

$$|f(x)|\leqslant |f_{n_0}(x)|+|f_n(x)-f(x)|<\varepsilon. \tag{7}$$

Thus by definition $\lim_{x\to\infty} f(x)=0$.