Math 118 Winter 2015 Homework 5 Solutions

Due Thursday Feb. 26 3pm in Assignment Box

Question 1. (5 pts) *Prove the following* ***by definition****.*

a) (2 pts) $\dfrac{1}{1+x^2}$ *is improperly integrable on* $(0,\infty)$.

b) (3 pts) $\tan x$ *is not improperly integrable on* $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Solution.

a) We have, for every $0<c<d<\infty$,

$$\int_c^d \frac{\mathrm{d}x}{1+x^2} = \arctan(d) - \arctan(c). \tag{1}$$

As $\lim_{d\to\infty}[\lim_{c\to 0+}[\arctan(d)-\arctan(c)]] = \frac{\pi}{2}$, $\dfrac{1}{1+x^2}$ is improperly integrable on $(0,\infty)$.

b) We have, for every $-\frac{\pi}{2}<c<d<\frac{\pi}{2}$,

$$\int_c^d \tan x\,\mathrm{d}x = \ln|\cos c| - \ln|\cos d|. \tag{2}$$

Now as $\lim_{c\to -\frac{\pi}{2}+} \ln|\cos c| = -\infty$ is not finite, $\tan x$ is not improperly integrable on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Question 2. (5 pts) *Let* $|f|$ *be improperly integrable on* (a,b) *and* g *be locally integrable on* (a,b)*. Further assume that* g *is bounded on* (a,b)*. Prove that* fg *is improperly integrable on* (a,b)*.*

Proof. Let $x_0\in(a,b)$. It suffices to prove the improper integrability of fg on $(a,x_0]$ and $[x_0,b)$. We prove it for $[x_0,b)$ and the other half of the proof is almost identical.

Let $d\in[x_0,b]$ be arbitrary. Consider

$$G(d) := \int_{x_0}^d f(x)\,g(x)\,\mathrm{d}x. \tag{3}$$

We will prove that $G(d)$ is Cauchy as $d\longrightarrow b-$.

Let $\varepsilon>0$ be arbitrary. As $|f(x)|$ is improperly integrable on (a,b), there is $d_0<b$ such that for all $b>d_2>d_1>d_0$,

$$\int_{d_1}^{d_2} |f(x)|\,\mathrm{d}x < \frac{\varepsilon}{M} \tag{4}$$

where $M\geqslant|g(x)|$ for all $x\in(a,b)$. Now let $d_2,d_1\in(d_0,b)$ be arbitrary. We have

$$|G(d_2)-G(d_1)| = \left|\int_{d_1}^{d_2} f(x)\,g(x)\,\mathrm{d}x\right| \leqslant \int_{d_1}^{d_2} |f(x)|\,M\,\mathrm{d}x < \varepsilon. \tag{5}$$

Therefore $G(d)$ is Cauchy and the improper integrability follows. □

Question 3. (5 pts) *Let* $f(x)$: $[1,\infty)$ *be positive and decreasing. Denote* $a_n := f(n)$ *for* $n\in\mathbb{N}$*. Prove*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Longleftrightarrow f(x) \text{ is improperly integrable on } (1,\infty). \tag{6}$$

Solution.

- $\Longrightarrow$. Let $M := \sum_{n=1}^{\infty} a_n$. Thus for all $n \in \mathbb{N}$, $a_1 + a_2 + \cdots + a_n < M$. Now as $f(x)$ is monotone, it is Riemann integrable on $[1, d]$ for every $d \in (1, \infty)$. Let $d \in (1, \infty)$ be arbitrary. There is $n \in \mathbb{N}$ such that $n > d$. We have

$$F(d) := \int_1^d f(x)\,\mathrm{d}x < \int_1^n f(x)\,\mathrm{d}x = \sum_{k=1}^{n-1} \int_k^{k+1} f(x)\,\mathrm{d}x \leqslant a_1 + \cdots + a_{n-1} < M. \quad (7)$$

thus $F(d)$ is increasing on $(1, \infty)$ with upper bound M, therefore $\lim_{d \to \infty} F(d)$ exists and is finite.

- $\Longleftarrow$. Let $n \in \mathbb{N}$ be arbitrary. We have

$$\sum_{k=1}^{n} a_k \leqslant a_1 + \int_1^n f(x)\,\mathrm{d}x < a_1 + \int_1^\infty f(x)\,\mathrm{d}x. \quad (8)$$

Thus $\sum_{k=1}^{n} a_k$ has a finite upper bound for all n. Since $a_n = f(n) > 0$ for all n, $\sum_{n=1}^{\infty} a_n$ converges.

QUESTION 4. (5 PTS + **5 PTS**) *Consider the function*

$$g(y) := \int_0^\infty e^{-xy} \frac{\sin x}{x}\,\mathrm{d}x. \quad (9)$$

a) (3 PTS) *Prove that $g(y)$ is defined for all $y > 0$. (Hint: Prove $\left|\frac{\sin x}{x}\right| \leqslant 1$)*

b) (2 PTS) *Prove that $\lim_{y \to \infty} g(y) = 0$.*

c) (**3 EXTRA PTS**) *Prove that $g(y)$ is differentiable on $(0, \infty)$ with $g'(y) = -\frac{1}{1+y^2}$.*

d) (**2 EXTRA PTS**) *Prove*

$$\lim_{y \to 0+} g(y) = \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x. \quad (10)$$

Note. *You should prove directly and not use theorems from multi-variable calculus.*

Proof.

a) Let $y > 0$ be arbitrary. We first prove $\left|\frac{\sin x}{x}\right| \leqslant 1$ for all $x \in [0, \infty)$. We apply MVT:

$$\left|\frac{\sin x}{x}\right| = \left|\frac{\sin x - \sin 0}{x - 0}\right| = |\cos c| \leqslant 1. \quad (11)$$

Thus we have

$$\left|e^{-xy} \frac{\sin x}{x}\right| \leqslant e^{-xy}. \quad (12)$$

Next we prove e^{-xy} is improperly integrable on $(0, \infty)$. As e^{-xy} is integrable on $[0, d]$ for all $d > 0$, we calculate

$$\int_0^d e^{-xy}\,\mathrm{d}x = \frac{1}{y} \int_0^d e^{-xy}\,\mathrm{d}(x\,y) = \frac{1 - e^{-dy}}{y} \longrightarrow \frac{1}{y} \text{ as } d \to \infty. \quad (13)$$

The conclusion now follows.

b) From the previous calculation we have

$$|g(y)| \leqslant \int_0^\infty e^{-xy}\,\mathrm{d}x = \frac{1}{y}. \tag{14}$$

Thus $\lim_{y\to\infty} g(y) = 0$ follows from Squeeze Theorem.

c) Let $y_0 \in (0,\infty)$ be arbitrary. Denote $A := -\int_0^\infty e^{-xy_0}\sin x\,\mathrm{d}x = -\frac{1}{1+y_0^2}$.

- Let $h \in (0, y_0)$. We have

$$\frac{g(y_0+h) - g(y_0)}{h} - A = \int_0^\infty \frac{e^{-xh} - 1 + x\,h}{x\,h}\, e^{-xy_0}\sin x\,\mathrm{d}x. \tag{15}$$

  By MVT we have

$$\left|\frac{e^{-xh} - 1}{x\,h}\right| = \left|\frac{e^{-xh} - e^{-0}}{x\,h - 0}\right| = |e^{-c}| \tag{16}$$

  for some $c \in (0, x\,h)$ which means

$$\left|\frac{e^{-xh} - 1 + x\,h}{x\,h}\right| \leqslant 2 \tag{17}$$

  for all $x > 0$.

  Next by Taylor expansion with Lagrange form of remainder we have

$$e^{-xh} = 1 - x\,h + \frac{e^{-c}}{2!}\, x^2\, h^2 \tag{18}$$

  for some $c \in (0, x\,h)$. Therefore

$$\left|\frac{e^{-xh} - 1 + x\,h}{x\,h}\right| \leqslant x\,h \tag{19}$$

  for all $x > 0$. Now we estiamte

$$\begin{aligned}\left|\frac{g(y_0+h) - g(y_0)}{h} - A\right| &\leqslant \left|\int_0^{h^{-1/2}} \frac{e^{-xh} - 1 + x\,h}{x\,h}\, e^{-xy_0}\sin x\,\mathrm{d}x\right| \\ &\quad + \left|\int_{h^{-1/2}}^\infty \frac{e^{-xh} - 1 + x\,h}{x\,h}\, e^{-xy_0}\sin x\,\mathrm{d}x\right| \\ &\leqslant h\int_0^{h^{-1/2}} x\, e^{-xy_0}\,\mathrm{d}x \\ &\quad + \int_{h^{-1/2}}^\infty 2\, e^{-xy_0}\,\mathrm{d}x. \end{aligned} \tag{20}$$

  It is clear that

$$\lim_{h\to 0+} \int_{h^{-1/2}}^\infty 2\, e^{-xy_0}\,\mathrm{d}x = 0. \tag{21}$$

  On the other hand, we have

$$\int_0^{h^{-1/2}} x\, e^{-xy_0}\,\mathrm{d}x = y_0^{-2}\int_0^{h^{-1/2}y_0} u\, e^{-u}\,\mathrm{d}u \leqslant y_0^{-2}\left(h^{-1/2}\, y_0 + 1\right). \tag{22}$$

  Therefore

$$\lim_{h\to 0+} h\int_0^{h^{-1/2}} x\, e^{-xy_0}\,\mathrm{d}x = 0. \tag{23}$$

Consequently

$$\lim_{h \to 0+} \frac{g(y_0 + h) - g(y_0)}{h} = A. \tag{24}$$

- Let $h \in \left(-\frac{y_0}{2}, 0\right)$. We have

$$\frac{g(y_0 + h) - g(y_0)}{h} - A = \int_0^\infty \frac{1 + x\,h\,e^{xh} - e^{xh}}{x\,h}\, e^{-x(y_0 + h)} \sin x \,\mathrm{d}x. \tag{25}$$

Similar to the $h > 0$ case, we have

$$\left|\frac{1 + x\,h\,e^{xh} - e^{xh}}{x\,h}\right| \leqslant 2, \qquad \left|\frac{1 + x\,h\,e^{xh} - e^{xh}}{x\,h}\right| \leqslant x\,|h|. \tag{26}$$

Furthermore we have

$$e^{-x(y_0 + h)} \leqslant e^{-x y_0/2}. \tag{27}$$

Now the proof proceeds similar to the case $h > 0$ and we conclude

$$\lim_{h \to 0-} \frac{g(y_0 + h) - g(y_0)}{h} = A. \tag{28}$$

d) Denote $I := \displaystyle\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$. We consider

$$g(y) - I = \int_0^\infty (e^{-xy} - 1)\,\frac{\sin x}{x}\,\mathrm{d}x. \tag{29}$$

Define $h(x) := \frac{e^{-xy} - 1}{x}$. Then

$$g(y) - I = \int_0^\infty h(x) \sin x \,\mathrm{d}x. \tag{30}$$

First notice that $\lim_{x \to \infty} h(x) = 0$ as $x \longrightarrow \infty$. Furthermore we calculate

$$h'(x) = \frac{-x\,y\,e^{-xy} + 1 - e^{-xy}}{x^2} = \frac{e^{-xy}}{x^2}\,[e^{xy} - 1 - x\,y] > 0 \text{ for all } x, y > 0. \tag{31}$$

Thus $h(x)$ increases from $-y$ to 0 as x runs from 0 to ∞.

We calculate

$$\begin{aligned} |g(y) - I| = \left|\int_0^\infty h(x) \sin x \,\mathrm{d}x\right| &= \left|-h(x) \cos x|_0^\infty + \int_0^\infty h'(x) \cos x \,\mathrm{d}x\right| \\ &\leqslant |h(0+)| + \int_0^\infty |h'(x)|\,\mathrm{d}x \\ &= 2\,|h(0+)| = 2\,y. \end{aligned} \tag{32}$$

Now it is obvious that $\lim_{y \to 0+} [g(y) - I] = 0$. □